

GLOBAL SOLUTIONS TO THE LANDAU-FERMI-DIRAC EQUATION

PAULO SAMPAIO

ABSTRACT. We establish a compactness result for solutions of a certain class of hypoelliptic equations. This result allows us to show the existence of global weak solutions to the non-homogeneous Landau-Fermi-Dirac equation with Coulomb potential.

CONTENTS

1. Introduction	1
2. Main result	3
3. A compactness result for the LFD equation	5
3.1. Velocity averages: compactness in time and position	6
3.2. Quasi-ellipticity of the diffusion matrix: compactness in velocity	9
3.3. Almost everywhere compactness	11
4. Cauchy theory for the approximated equation	14
5. The existence theorem	27
References	35

1. INTRODUCTION

Kinetic theory aims to describe a gas by a probability density function $f = f(t, x, v)$. That is, for a time t , the probability of finding a particle in the region dx with velocity in the range dv is $f(t, x, v)dx dv$.

The most famous description for f is given by the Boltzmann equation. A particularly interesting case is the one with hot gases of charged particles, such as plasmas. In this regime, grazing collisions prevail and this limit gives rise to another model, called Landau equation.

If one considers fermions (particles described by Fermi-Dirac statistics, such as electrons), then quantum effects must be taken into account and this gives rise to the Boltzmann-Fermi-Dirac (BFD) and Landau-Fermi-Dirac (LFD) equations.

In this paper, we study the Cauchy problem for the LFD equation, i.e. given an initial data f_0 , find a function f such that

$$(1) \quad \begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left(\int a(v - v_*) (f_*(1 - \delta f_*) \nabla_v f - f(1 - \delta f) \nabla_{v_*} f_*) dv_* \right) \\ f|_{t=0} = f_0 \end{cases}$$

Here, we have used the common notation $f = f(t, x, v)$ and $f_* = f(t, x, v_*)$. The collision kernel $a = (a_{ij})_{ij}$ has the form

$$(2) \quad a(z) = \Gamma(|z|) \left(I - \frac{z \otimes z}{|z|^2} \right),$$

where the cross section $\Gamma(|\cdot|)$ is a function that depends on the type of interaction between the particles.

The parameter $\delta > 0$ is proportional to the cube of Planck's constant \hbar and thus measures the quantum effects modeled by the equation. Accordingly, we see that in the limit case $\delta = 0$ we recover the (classical) Landau equation.

If the particle interaction is held by a power law potential, then

$$\Gamma(|z|) = |z|^{\gamma+2}, \text{ with } -3 \leq \gamma \leq 1.$$

In the literature, we find that the nomenclature of the potential varies depending on the range in which the γ are considered, so generally if $0 < \gamma \leq 1$ we say that it is a hard potential, if $\gamma < 0$ then it is a (moderately) soft potential, if $\gamma < -2$ then it's a very soft potential and if $\gamma = -3$ then it is a Coulomb potential.

As for the classical Landau equation, this form of the matrix a allows us to deduce, at least formally, that

$$\frac{d}{dt} \iint f(t) \varphi \, dx dv = 0$$

if $\varphi = 1, v_i, |v|^2$ or $|x - vt|^2$, which physically corresponds to the conservation of mass, linear momentum, kinetic energy and moment of inertia, respectively.

Also, if we define the (quantum) entropy

$$S(t) = \frac{1}{\delta} \iint \delta f \log(\delta f) + (1 - \delta f) \log(1 - \delta f) \, dx dv,$$

it follows that, at least formally, this quantity should be decreasing in time.

These conservation laws, as well as the entropy decay leads us to assume that physically acceptable solutions to this equation must satisfy the uniform bound

$$(3) \quad \sup_{t>0} \iint f(1 + |x - vt|^2 + |v|^2 + \log f) \, dx dv < +\infty.$$

provided this bound holds at time $t = 0$. In the study of kinetic equations, these are called *a priori* bounds.

The case we will be interested in this paper, which is also the most physically interesting case, is the one where the particles interact through a Coulomb potential, and we therefore usually think $\gamma = -3$, but the whole argument is also valid for very soft potentials. This is, however, the least understood case, due to the singularity at $z = 0$.

For the Landau equation, the bounds implied by (3) are not sufficient for making sense out of a solution to the equation, even in the sense of distributions, because the collision integral is not well defined in this case. To get around this problem, in 1994 Lions [14] defined the notion of a *renormalized solution*, in the same spirit as his earlier work with Di Perna for the Boltzmann equation [7].

The main technique in this case is to derive a *sequential stability* result for solutions of the equation. Then, constructing smooth solutions to a carefully chosen approximate equation, this compactness theorem allows us to pass to the limit, thus showing existence for the desired equation.

For the Landau equation, even though we have a compactness result for its renormalized solutions (see Lions [14]), we cannot pass the renormalized equation directly to the limit and a defect measure appears, as shown by Villani [15], thus leading to an even weaker sense of solution.

Fermions, on the other hand, obey the *Pauli Exclusion Principle*, which states that no two particles can occupy the same quantum state at the same instant. Mathematically, this means that

$$(4) \quad 0 \leq f \leq \frac{1}{\delta},$$

and thus the Pauli principle gives us an extra L^∞ bound for the solutions. This implies that the collision integral is well defined even for the Coulomb case, thus in the quantum case we don't expect the need for renormalized solutions or the appearance of defect measures.

The Cauchy problem for quantum fluids goes back as far as Dolbeaut [8], who showed existence and uniqueness of solutions to the BFD equation with cutoff satisfying the conservation laws and decay of entropy. Afterwards, Alexandre [2] then showed the existence of solutions to the BFD equation without the cutoff hypothesis.

The first significant study of the LFD equation was conducted Bagland [4] in 2004, who showed the existence of solutions for hard potentials in the homogeneous case (where f does not depend on x). Recently, multiple results on the Cauchy problem for homogeneous LFD has been proven. Alonso, Bagland, Desvillettes and Lods [3] have shown existence for soft potentials and Golding, Gualdani and Zamponi [9] have shown existence and smoothness for a Coulomb potential.

The inhomogeneous case of the LFD equation, however, remains poorly understood and to the author's knowledge, there are no results on the Cauchy problem for this equation.

In this paper, we will show existence of (weak) solutions to the LFD equation in the Coulomb case, hugely inspired by the works of Lions [14] and Villani [15].

2. MAIN RESULT

Throughout the paper, we will use the repeated index summation convention and we will note $\mathbb{R}_{x,v}^{2N} = \mathbb{R}_x^N \times \mathbb{R}_v^N$ the phase space.

Relabeling δf as f and a_{ij} as a_{ij}/δ , one can rewrite the equation as

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{\partial}{\partial v_i} \left(\bar{a}_{ij} \frac{\partial f}{\partial v_j} - \bar{b}_i f(1-f) \right),$$

where $\bar{a}_{ij} = a_{ij} *_{\nu} (f(1-f))$ and $\bar{b}_i = \frac{\partial a_{ij}}{\partial v_j} *_{\nu} f$. This way we can, without loss of generality, suppose $\delta = 1$.

Throughout the paper we will assume that the cross section Γ satisfies

$$(5) \quad \forall R > 0 \text{ there exists a } K_R > 0 \text{ such that } \Gamma(|z|) \geq K_R, \forall |z| \leq R,$$

which will guarantee some ellipticity for the collision kernel and also that it has integrability

$$(6) \quad \Gamma(|z|) \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N),$$

for some $r > \frac{N}{N-1}$, which means that $\Gamma(|z|)$ can be decomposed as the sum of a function in $L^r(\mathbb{R}^N)$ with a function in $L^\infty(\mathbb{R}^N)$.

It is worth noting that the properties (5) and (6) cover the cases discussed above of very smooth potentials and, most importantly, Coulomb potentials. Other cases can be treated using variations of the techniques discussed here.

We now state explicitly what we mean by a solution of the equations we will deal in this paper.

Definition 1. Let $T > 0$ and $\bar{A}_{ij}, \bar{B}_i, \bar{C}, \bar{D} \in L_{loc}^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$. A function $f = f(t, x, v)$ in $L_{loc}^1((0, \infty) \times \mathbb{R}_{x,v}^{2N}) \cap C((0, \infty); \mathcal{D}'(\mathbb{R}_{x,v}^{2N}))$ is called a (global) weak solution of the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{\partial}{\partial v_i} \left(\bar{A}_{ij} \frac{\partial f}{\partial v_j} + \bar{B}_i f \right) + \bar{C} f + \bar{D}$$

with initial data f_0 if it solves the equation in $\mathcal{D}'((0, \infty) \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$, that is, for every test function $\varphi \in \mathcal{D}((0, \infty) \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$, we have

$$\begin{aligned}
& - \int_0^\infty \iint f \frac{\partial \varphi}{\partial t} dx dv dt - \int_0^\infty \iint f v_i \frac{\partial \varphi}{\partial x_i} dx dv dt = \int_0^\infty \iint \bar{A}_{ij} f \frac{\partial^2 \varphi}{\partial v_i \partial v_j} dx dv dt \\
& + \int_0^\infty \iint \left(\frac{\partial \bar{A}_{ij}}{\partial v_j} f + \bar{B}_i f \right) \frac{\partial \varphi}{\partial v_i} dx dv dt + \int_0^\infty \iint (\bar{C} f + \bar{D}) \varphi dx dv dt
\end{aligned}$$

and also for every $\psi \in \mathcal{D}(\mathbb{R}_x^N \times \mathbb{R}_v^N)$, we have

$$\iint f(t) \psi dx dv \xrightarrow{t \rightarrow 0^+} \iint f_0 \psi dx dv.$$

Notating $L_2^1(\mathbb{R}_{x,v}^{2N})$ the set of functions f such that

$$\iint_{\mathbb{R}_{x,v}^{2N}} |f(x, v)| (1 + |x|^2 + |v|^2) dx dv < +\infty,$$

our main result then reads

Theorem 1. *Let $N \geq 2$ and $f_0 \in L_2^1(\mathbb{R}_{x,v}^{2N})$ be such that $0 \leq f_0 \leq 1$. There exists a weak solution $f \in C((0, \infty); \mathcal{D}'(\mathbb{R}_{x,v}^{2N})) \cap L^\infty((0, \infty); L_2^1(\mathbb{R}_{x,v}^{2N}))$ to the LFD equation with initial data f_0 , in the sense of Definition 1, which satisfies, for a.e. $t \geq 0$,*

1) *Pauli exclusion principle:*

$$0 \leq f(t) \leq 1.$$

2) *Conservation of mass and linear momentum:*

$$\iint f(t, x, v) dx dv = \iint f_0(x, v) dx dv,$$

$$\iint f(t, x, v) v_i dx dv = \iint f_0(x, v) v_i dx dv \quad \forall i \in \{1, \dots, N\}.$$

3) *Decay of kinetic energy and moment of inertia:*

$$\iint f(t, x, v) |v|^2 dx dv \leq \iint f_0(x, v) |v|^2 dx dv,$$

$$\iint f(t, x, v) |x - tv|^2 dx dv \leq \iint f_0(x, v) |x - tv|^2 dx dv.$$

4) *Entropy inequality:*

$$\iint s(t, x, v) dx dv + \int_0^t \iint d(\tau, x, v) dx dv d\tau \leq \iint s(0, x, v) dx dv,$$

where

$$s(t, x, v) = f \log f + (1 - f) \log(1 - f)$$

is the quantum entropy and

$$(7) \quad d(t, x, v) = \int a(v - v_*) f(1 - f) f_*(1 - f_*) \left| \frac{\nabla_v f}{f(1 - f)} - \frac{\nabla_{v_*} f_*}{f_*(1 - f_*)} \right|^{\otimes 2} dv_*,$$

is the quantum entropy dissipation.

Remark 1. *Given that weak solutions may have no regularity in v , one might wonder what expression (7) above means. In the proof of the entropy inequality, we will see that the integral $\int_0^t \iint d(\tau, x, v) dx dv d\tau$ is a notation for the $L^2((0, t) \times \mathbb{R}_x^N \times \mathbb{R}_v^N \times \mathbb{R}_{v_*}^N)$ norm of*

$$\begin{aligned} & \operatorname{div}_v \left[\sqrt{a(v - v_*)} \sqrt{f_*(1 - f_*)} \arcsin \sqrt{f} \right] - \operatorname{div}_{v_*} \left[\sqrt{a(v - v_*)} \sqrt{f(1 - f)} \arcsin \sqrt{f_*} \right] \\ & - \operatorname{div}_v \left(\sqrt{a(v - v_*)} \right) \sqrt{f_*(1 - f_*)} \arcsin \sqrt{f} + \operatorname{div}_{v_*} \left(\sqrt{a(v - v_*)} \right) \sqrt{f(1 - f)} \arcsin \sqrt{f_*}, \end{aligned}$$

where the divergence of a matrix A is defined as $\operatorname{div}_v A = \sum_j \frac{\partial A_{ij}}{\partial v_j}$. The motivation for using the expression (7) is that if f has some regularity in v , then the two expressions are equal, which we will see later in the proof.

We have divided the text into three sections. In Section 3, we will prove a compactness result for the solutions of LFD-like equations, which involves studying the parabolic and transport parts of this equation separately in order to achieve compactness for the sequence of solutions.

Moving on to Section 4, we will show the existence of solutions to an approximate equation. We will define the approximation to be used and construct solutions for it that will exhibit better integrability and regularity, which we will use to show that they satisfy the conservation laws (mass, momentum, energy, ...) in an approximate sense.

Lastly, in Section 5 we construct a sequence of regularized solutions with progressively weaker regularization. This approximation scheme will be compatible with the compactness theorem of Section 3, which will allow us to pass to the limit and show that it is indeed a solution of the LFD equation obeying the conservation laws and inequalities.

3. A COMPACTNESS RESULT FOR THE LFD EQUATION

Throughout the whole paper, we will use the notation \int_{xv} for the integral $\iint_{\mathbb{R}_{x,v}^{2N}} dx dv$. In this section, our main goal is to prove the following compactness result for LFD-like equations. We will use the space $L_{loc}^2((0, \infty) \times \mathbb{R}_x^N; H_{loc}^1(\mathbb{R}_v^N))$ of $L_{loc}^2((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ functions whose v derivatives in the sense of distributions $\frac{\partial f}{\partial v_i}$ can be identified to $L_{loc}^2((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ functions.

Proposition 1. *Let $\bar{A}^n, \frac{\partial \bar{A}_{ij}^n}{\partial v_j}, \bar{B}_i^n, \frac{\partial \bar{B}_i^n}{\partial v_i}$ be $L_{loc}^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ functions and $(g^n)_n$ be a sequence of weak solutions in $L_{loc}^2((0, \infty) \times \mathbb{R}_x^N; H_{loc}^1(\mathbb{R}_v^N))$ to*

$$(8) \quad \begin{cases} \frac{\partial g^n}{\partial t} + v \cdot \nabla_x g^n = \operatorname{div}_v \left[\bar{A}^n \nabla_v g^n + \bar{B}^n g^n (1 - g^n) \right] \\ g^n|_{t=0} = g_0^n, \end{cases}$$

in the sense of Definition 1, such that, for every $\eta \in \mathbb{R}^N$ and $R, R' > 0$ there exists some $\nu = \nu(R, R') > 0$ such that

$$(9) \quad \bar{A}_{ij}^n \eta_i \eta_j \geq \int_{|v_*| \leq R'} \nu \left[1 - \left(\frac{v - v_*}{|v - v_*|} \cdot \frac{\eta}{|\eta|} \right)^2 \right] |\eta|^2 g_*^n (1 - g_*^n) dv_*, \quad \forall v \in \mathbb{R}_v^N, |v| \leq R$$

Suppose moreover that $(g^n)_n$ satisfies

$$\begin{aligned} & \sup_{t>0} \int_{xv} g^n (1 + |x - vt|^2 + |v|^2) \leq C, \\ & 0 \leq g^n \leq 1. \end{aligned}$$

If $\overline{A}^n, \frac{\partial \overline{A}_{ij}^n}{\partial v_j}, \overline{B}_i^n, \frac{\partial \overline{B}_i^n}{\partial v_i}$ are uniformly bounded in $L_{loc}^1((0, \infty) \times \mathbb{R}_{x,v}^{2N})$, then the sequence $(g^n)_n$ is compact in $L_{loc}^1((0, \infty); L^1(\mathbb{R}_{x,v}^{2N}))$.

Our approach to proving this proposition will be to achieve compactness bit by bit. First, we use the transport structure of the equation to achieve compactness in space-time of averages in velocity, which are the now famous *velocity averaging* results. In a second step, we study the parabolic part of the equation to deduce a certain compactness in the velocity variable. Finally, we unify this two compactness properties with a proposition, leading the "full" compactness, in all three variables.

3.1. Velocity averages: compactness in time and position. Velocity averaging is a very common technique in the study of transport equations, and in particular kinetic equations, which allows us to find a certain type of compactness for the velocity averages of the solutions of that equation.

The main idea is that if $f = f(t, x, v)$ is a solution of the transport equation $\partial_t f + v \cdot \nabla_x f = g$, then the velocity average $\int f \varphi(v) dv$, where $\varphi \in C_c^\infty(\mathbb{R}_v^N)$, is shown to be more regular than f and g themselves. For a sequence of solutions f^n , this translates into compactness for the averaged sequence $\int f^n \varphi(v) dv$.

The first results in this direction were shown independently in the mid-80's by Agoshkov [1] and Golse, Perthame and Sentis in [10] in an L^2 setting. We will use a generalized version taken from [5], reproduced here for the reader's convenience.

Theorem 2 (Theorem 1.1.8 from [5]). *Let Ω be an open set of $\mathbb{R}_t \times \mathbb{R}_x^N$, $a : \mathbb{R}^M \rightarrow \mathbb{R}^N$ verify $\partial^\alpha a \in L_{loc}^\infty$ for $|\alpha| \leq m$, $m \in \mathbb{N}$, and*

$$\forall \sigma \in S^{N-1} \forall u \in \mathbb{R} \ |v; a(v) \cdot \sigma = u| = 0.$$

Let f_n be a bounded sequence in $L_{loc}^p(\Omega \times \mathbb{R}^M)$ with $p > 1$ verifying

$$\partial_t f_n + \operatorname{div}_x a(v) f_n = \sum_{|\alpha| \leq m} \partial_v^\alpha g_n^{(\alpha)} \text{ in } \Omega \times \mathbb{R}^M$$

where the $g_n^{(\alpha)}$ are locally bounded in the space of measures $\mathcal{M}_{loc}(\Omega \times \mathbb{R}^M)$.

Then, for any function $\psi \in C_c^\infty(\mathbb{R}^M)$,

$$\rho_\psi^n(t, x) = \int f_n(t, x, v) \psi(v) dv$$

is compact in $L_{loc}^q(\Omega)$ for any $q < p$.

For reasons that will become clear when we discuss the parabolic part of the equation in depth, the key quantity to show compactness of solutions of this equation are not the solutions g^n themselves, but the quantity $g^n(1 - g^n)$.

Lemma 1. *Let $(g^n)_n$ be a sequence of solutions to (8) as in Proposition 1. For every $\varphi \in C_c^\infty(\mathbb{R}_v^N)$, the sequence of velocity averages*

$$\int g^n(1 - g^n) \varphi dv$$

is compact in $L_{loc}^1((0, \infty) \times \mathbb{R}_x^N)$ and moreover, for every $R, T > 0$, we have

$$(10) \quad \int_0^T \int_{xv} \overline{A}_{ij}^n \frac{\partial g^n}{\partial v_i} \frac{\partial g^n}{\partial v_j} dt \leq C_{T,R},$$

for some constant $C_{T,R} > 0$, independent of n .

Proof. Let us notate $G^n = g^n(1 - g^n)$. Throughout this proof, we will use the notation \int_{txv} for the integral $\int_0^\infty \iint_{\mathbb{R}_{x,v}^{2N}} dx dv dt$.

Theorem 2 allows us to deduce the compactness of the velocity averages of g^n , after rewriting the right-hand side of the equation. However, we are unable to deduce the compactness of velocity averages of G^n from the compactness of the velocity averages g^n alone. The alternative is then to write the equation for G^n and apply Theorem 2 to this new one. The process of passing from (8) to an equation for G^n is not direct, since such a process would require applying a product rule to $g^n(1 - g^n)$, which cannot be justified due to the low regularity in the variables t and x .

The way to overcome this limitation is to consider approximations of the g^n functions. Let then $\chi = \chi(t, x)$ be a $C_c^\infty((0, \infty) \times \mathbb{R}_x^N)$ positive function such that $\int \chi dt dx = 1$, consider the mollifying sequence $\chi_m(t, x) = m^N \chi(mt, mx)$ and define $g_m^n = \chi_m * g^n$. We thus have $g_m^n \rightarrow g^n$ in $L_{loc}^2((0, \infty) \times \mathbb{R}_{x,v}^{2N})$. Take the equation in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{x,v}^{2N})$,

$$\frac{\partial g^n}{\partial t} + v \cdot \nabla_x g^n = \frac{\partial}{\partial v_i} \left(\overline{A}_{ij}^n \frac{\partial g^n}{\partial v_j} + \overline{B}_i^n g^n(1 - g^n) \right).$$

Applying the approximations we have, commuting convolutions with derivatives

$$\frac{\partial g_m^n}{\partial t} + v \cdot \nabla_x g_m^n = \frac{\partial}{\partial v_i} \left[\chi_m * \left(\overline{A}_{ij}^n \frac{\partial g^n}{\partial v_j} \right) \right] + \chi_m * \left(\overline{B}_i^n g^n(1 - g^n) \right).$$

Multiplying both sides of the equation by $(1 - 2g_m^n)$ and noting $G_m^n \equiv g_m^n(1 - g_m^n)$, we have

$$\begin{aligned} \frac{\partial G_m^n}{\partial t} + v \cdot \nabla_x G_m^n &= \frac{\partial}{\partial v_i} \left[\chi_m * \left(\overline{A}_{ij}^n \frac{\partial g^n}{\partial v_j} \right) (1 - 2g_m^n) \right] + 2\chi_m * \left(\overline{A}_{ij}^n \frac{\partial g^n}{\partial v_j} \right) \frac{\partial g_m^n}{\partial v_i} \\ &\quad + \frac{\partial}{\partial v_i} \left[\chi_m * \left(\overline{B}_i^n g^n(1 - g^n) \right) (1 - 2g_m^n) \right] + 2\chi_m * \left(\overline{B}_i^n g^n(1 - g^n) \right) \frac{\partial g_m^n}{\partial v_i}. \end{aligned}$$

We multiply this equation on both sides by $\varphi \in \mathcal{D}((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ and integrate by parts to find a formulation in the sense of distributions

$$\begin{aligned} - \int_{txv} G_m^n \frac{\partial \varphi}{\partial t} - \int_{txv} G_m^n v_i \frac{\partial \varphi}{\partial x_i} &= - \int_{txv} \chi_m * \left(\overline{A}_{ij}^n \frac{\partial g^n}{\partial v_j} \right) (1 - 2g_m^n) \frac{\partial \varphi}{\partial v_i} \\ &\quad + 2 \int_{txv} \chi_m * \left(\overline{A}_{ij}^n \frac{\partial g^n}{\partial v_j} \right) \frac{\partial g_m^n}{\partial v_i} \varphi - \int_{txv} \chi_m * \left(\overline{B}_i^n g^n(1 - g^n) \right) (1 - 2g_m^n) \frac{\partial \varphi}{\partial v_i} \\ &\quad + 2 \int_{txv} \chi_m * \left(\overline{B}_i^n g^n(1 - g^n) \right) \frac{\partial g_m^n}{\partial v_i} \varphi \end{aligned}$$

and by the standard theorems of approximation by convolution, we have that, passing to the limit $m \rightarrow \infty$,

$$\begin{aligned} - \int_{txv} G^n \frac{\partial \varphi}{\partial t} - \int_{txv} G^n v_i \frac{\partial \varphi}{\partial x_i} &= - \int_{txv} \overline{A}_{ij}^n \frac{\partial g^n}{\partial v_j} (1 - 2g^n) \frac{\partial \varphi}{\partial v_i} + 2 \int_{txv} \overline{A}_{ij}^n \frac{\partial g^n}{\partial v_j} \frac{\partial g^n}{\partial v_i} \varphi \\ &\quad - \int_{txv} \overline{B}_i^n g^n(1 - g^n) (1 - 2g^n) \frac{\partial \varphi}{\partial v_i} + 2 \int_{txv} \overline{B}_i^n g^n(1 - g^n) \frac{\partial g^n}{\partial v_i} \varphi. \end{aligned}$$

Now, since we have $g^n, 1 - g^n \in L_{loc}^2((0, \infty) \times \mathbb{R}_x^N; H_{loc}^1(\mathbb{R}_v^N))$, the product rule is valid. We have $\frac{\partial(g^n(1 - g^n))}{\partial v_k} \in L_{loc}^1((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ for every $k = 1, 2, \dots, N$ and

$$\frac{\partial(g^n(1 - g^n))}{\partial v_k} = (1 - 2g^n) \frac{\partial g^n}{\partial v_k}.$$

Let $\psi_m = \psi_m(t, x, v)$ be a mollifying sequence and consider $(\bar{A}_{ij}^n)_m \equiv \psi_m * \bar{A}_{ij}^n$. This way, we have $(\bar{A}_{ij}^n)_m \rightarrow \bar{A}_{ij}^n$ and $\frac{\partial(\bar{A}_{ij}^n)_m}{\partial v_k} \rightarrow \frac{\partial \bar{A}_{ij}^n}{\partial v_k}$ in $L_{loc}^p((0, \infty) \times \mathbb{R}_{x,v}^{2N})$, $p < \infty$. Since $(\bar{A}_{ij}^n)_m \frac{\partial \varphi}{\partial v_i} \in C_c^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$, we then have, by the definition of weak derivative,

$$\begin{aligned} \int_{txv} (\bar{A}_{ij}^n)_m \frac{\partial g^n}{\partial v_j} (1 - 2g^n) \frac{\partial \varphi}{\partial v_i} &= \int_{txv} \frac{\partial(g^n(1 - g^n))}{\partial v_j} (\bar{A}_{ij}^n)_m \frac{\partial \varphi}{\partial v_i} \\ &= - \int_{txv} g^n(1 - g^n) \frac{\partial}{\partial v_j} \left((\bar{A}_{ij}^n)_m \frac{\partial \varphi}{\partial v_i} \right) \\ &= - \int_{txv} g^n(1 - g^n) \frac{\partial(\bar{A}_{ij}^n)_m}{\partial v_j} \frac{\partial \varphi}{\partial v_i} - \int_{txv} g^n(1 - g^n) (\bar{A}_{ij}^n)_m \frac{\partial^2 \varphi}{\partial v_i \partial v_j} \end{aligned}$$

thus passing to the limit $m \rightarrow \infty$, we have

$$\int_{txv} \bar{A}_{ij}^n \frac{\partial g^n}{\partial v_j} (1 - 2g^n) \frac{\partial \varphi}{\partial v_i} = - \int_{txv} g^n(1 - g^n) \frac{\partial \bar{A}_{ij}^n}{\partial v_j} \frac{\partial \varphi}{\partial v_i} - \int_{txv} g^n(1 - g^n) \bar{A}_{ij}^n \frac{\partial^2 \varphi}{\partial v_i \partial v_j}$$

and similarly, one can show that

$$\int_{txv} \bar{B}_i^n g^n(1 - g^n) \frac{\partial g^n}{\partial v_i} \varphi = - \int_{txv} \frac{\partial \bar{B}_i^n}{\partial v_i} \left(\frac{(g^n)^2}{2} - \frac{(g^n)^3}{3} \right) \varphi - \int_{txv} \bar{B}_i^n \left(\frac{(g^n)^2}{2} - \frac{(g^n)^3}{3} \right) \frac{\partial \varphi}{\partial v_i},$$

meaning the equation can be rewritten as

$$\begin{aligned} (11) \quad & - \int_{txv} G^n \frac{\partial \varphi}{\partial t} - \int_{txv} G^n v_i \frac{\partial \varphi}{\partial x_i} = \int_{txv} \bar{A}_{ij}^n g^n(1 - g^n) \frac{\partial^2 \varphi}{\partial v_i \partial v_j} + \int_{txv} \frac{\partial \bar{A}_{ij}^n}{\partial v_j} g^n(1 - g^n) \frac{\partial \varphi}{\partial v_i} \\ & + 2 \int_{txv} \bar{A}_{ij}^n \frac{\partial g^n}{\partial v_i} \frac{\partial g^n}{\partial v_j} \varphi - \int_{txv} \bar{B}_i^n g^n(1 - g^n)(1 - 2g^n) \frac{\partial \varphi}{\partial v_i} \\ & - 2 \int_{txv} \frac{\partial \bar{B}_i^n}{\partial v_i} \left(\frac{(g^n)^2}{2} - \frac{(g^n)^3}{3} \right) \varphi - 2 \int_{txv} \bar{B}_i^n \left(\frac{(g^n)^2}{2} - \frac{(g^n)^3}{3} \right) \frac{\partial \varphi}{\partial v_i}. \end{aligned}$$

Then, rearranging the equation and using that $0 \leq g^n \leq 1$, we have

$$\begin{aligned} \int_{txv} \bar{A}_{ij}^n \frac{\partial g^n}{\partial v_i} \frac{\partial g^n}{\partial v_j} \varphi dx dv &\leq \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^1} + \|v \cdot \nabla_x \varphi\|_{L^1} + \left\| \frac{\partial \bar{A}_{ij}^n}{\partial v_j} \right\|_{L^1(\text{supp } \varphi)} \left\| \frac{\partial \varphi}{\partial v_i} \right\|_{L^\infty} \\ &+ \left\| \bar{A}_{ij}^n \right\|_{L^1(\text{supp } \varphi)} \left\| \frac{\partial^2 \varphi}{\partial v_i \partial v_j} \right\|_{L^\infty} + 3 \left\| \bar{B}_i^n \right\|_{L^1(\text{supp } \varphi)} \left\| \frac{\partial \varphi}{\partial v_i} \right\|_{L^\infty} \\ &+ 2 \left\| \frac{\partial \bar{B}_i^n}{\partial v_i} \right\|_{L^1(\text{supp } \varphi)} \|\varphi\|_{L^\infty} \end{aligned}$$

and thus, given the uniform boundedness of the coefficients, the inequality (10) follows by taking φ such that $\varphi \equiv 1$ in $[0, T] \times B_R \times B_R$. In particular, this implies that $\bar{A}_{ij}^n \frac{\partial g^n}{\partial v_i} \frac{\partial g^n}{\partial v_j}$ is uniformly bounded in $L_{loc}^1((0, T) \times \mathbb{R}_{x,v}^{2N})$.

Now we prove the compactness of the velocity averages. Note that the equality (11) means we have

$$\begin{aligned} \frac{\partial G^n}{\partial t} + v_i \frac{\partial G^n}{\partial x_i} &= \frac{\partial^2}{\partial v_i \partial v_j} \left[\bar{A}_{ij}^n g^n (1 - g^n) \right] \\ &+ \frac{\partial}{\partial v_i} \left[\bar{B}_i^n g^n (1 - g^n) (1 - 2g^n) - \frac{\partial \bar{A}_{ij}^n}{\partial v_j} g^n (1 - g^n) + 2\bar{B}_i^n \left(\frac{(g^n)^2}{2} - \frac{(g^n)^3}{3} \right) \right] \\ &+ 2\bar{A}_{ij}^n \frac{\partial g^n}{\partial v_i} \frac{\partial g^n}{\partial v_j} - 2\frac{\partial \bar{B}_i^n}{\partial v_i} \left(\frac{(g^n)^2}{2} - \frac{(g^n)^3}{3} \right) \end{aligned}$$

in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{x,v}^{2N})$. Since we have that $0 \leq g^n \leq 1$ for all n , it follows that

$$\begin{aligned} |\bar{A}_{ij}^n g^n (1 - g^n)| &\leq |\bar{A}_{ij}^n| \\ \left| \bar{B}_i^n g^n (1 - g^n) (1 - 2g^n) - \frac{\partial \bar{A}_{ij}^n}{\partial v_j} g^n (1 - g^n) + 2\bar{B}_i^n \left(\frac{(g^n)^2}{2} - \frac{(g^n)^3}{3} \right) \right| &\leq 3|\bar{B}_i^n| + \left| \frac{\partial \bar{A}_{ij}^n}{\partial v_j} \right| \\ \left| \bar{A}_{ij}^n \frac{\partial g^n}{\partial v_i} \frac{\partial g^n}{\partial v_j} - \frac{\partial \bar{B}_i^n}{\partial v_i} \left(\frac{(g^n)^2}{2} - \frac{(g^n)^3}{3} \right) \right| &\leq \bar{A}_{ij}^n \frac{\partial g^n}{\partial v_i} \frac{\partial g^n}{\partial v_j} + \left| \frac{\partial \bar{B}_i^n}{\partial v_i} \right|, \end{aligned}$$

implying each of this sequences of functions is uniformly bounded in $L_{loc}^1((0, \infty) \times \mathbb{R}_{x,v}^{2N})$. Thus, Theorem 2 applies and we have that the velocity averages of $g^n(1 - g^n)$ are compact. \square

3.2. Quasi-ellipticity of the diffusion matrix: compactness in velocity. Here we will begin to explore the parabolic part of the equation in order to find compactness in the v derivatives of g^n .

The technique here is hugely inspired by the parabolic case. Indeed, if the equations (8) were uniformly parabolic, then the matrices \bar{A}^n would be uniformly elliptic, implying there exists some $\theta > 0$ such that $\bar{A}_{ij}^n \eta_i \eta_j \geq \theta |\eta|^2$ for every $\eta \in \mathbb{R}^N$.

This, together with inequality (10), gives directly a bound for the v derivatives of the sequence g^n , which would then imply compactness in the v variable. Thus, one may seek to show some kind of ellipticity estimate uniform in n for the sequence $\bar{a}^n = (\bar{a}_{ij}^n)_{ij} = (a_{ij} *_v f(1 - f))_{ij}$ of the LFD equation.

For the homogeneous case Bagland [4] has shown that the a priori bounds for entropy, mass and kinetic energy can control the size of $g(1 - g)$ from below, allowing us to prove an ellipticity estimate using the techniques from Desvillettes-Villani [6]. However, this technique doesn't translate to the non-homogeneous case, and we weren't able to prove this type of ellipticity using just the a priori estimates. Following Lions [14], we will prove a weaker, more specific, kind of ellipticity that turns out to be enough for reaching compactness.

Let then $\eta \in \mathbb{R}^N$. The estimate (9) gives us that

$$\bar{A}_{ij}^n \eta_i \eta_j \geq \int_{|v_*| \leq R'} \nu \left[1 - \left(\frac{v - v_*}{|v - v_*|} \cdot \frac{\eta}{|\eta|} \right)^2 \right] |\eta|^2 g_*^n (1 - g_*^n) dv_*, \quad \forall v \in \mathbb{R}^N, |v| \leq R$$

Here we see that there are essentially two problems preventing the right-hand side from being elliptic: an angular problem, when we have a large angle between $v - v_*$ and η and a quantum vacuum one, when the functions $g(1 - g)$ are too small.

The first one is purely geometrical and can be dealt with truncation arguments. The second one, on the other hand, depends entirely on the behavior of the functions $g^n(1 - g^n)$.

Define $G^n \equiv g^n(1 - g^n)$. Intuitively, we expect that for n sufficiently large, we have \bar{A}_{ij}^n elliptic in the regions where G^n is strictly positive. Asymptotically, one then expects the uniform elliptical

behavior of the sequence \bar{A}_{ij}^n to be described by some limit of the G^n , say, G . The following result establishes that, in the regions of space-time where G is not a vacuum (i.e., the density ρ is strictly greater than some $\alpha > 0$), then we have some asymptotically uniform ellipticity, with the constant depending on α , up to some set of small measure.

Lemma 2. *Fix $T, R > 0$. Let $G^n \equiv g^n(1 - g^n)$ be a sequence and G a function of $L^\infty((0, T) \times \mathbb{R}_{x,v}^{2N})$ such that*

$$\int G^n \varphi dv \rightarrow \int G \varphi dv \text{ in } L^1([0, T] \times B_R)$$

for every $\varphi \in L^1(\mathbb{R}_v^N)$. Moreover, let

$$K_\alpha \equiv \{(t, x) \in [0, T] \times B_R : \rho(t, x) > \alpha\},$$

where $\rho(t, x) = \int G dv$ and let \bar{A}^n be a sequence satisfying (9). Then, for every $\alpha, \varepsilon > 0$ there exists a measurable set $|E| < \varepsilon$ such that

$$\bar{A}_{ij}^n(t, x, v) \eta_i \eta_j \geq C(\alpha, \varepsilon) |\eta|^2,$$

for $(x, t, v, \eta) \in (K_\alpha \cap E^c) \times B_R \times \mathbb{R}^N$ and $n \geq n_0(\varepsilon, \alpha)$.

Proof. We can suppose, without loss of generality, that $\eta \in \mathbb{S}^{N-1}$.

To deal with the angular problem impeaching ellipticity, we introduce the parameter $\mu \in [0, 1]$ and define the following set, which imposes the R' constraint and truncates the "bad" angles,

$$V(v, \mu, \eta) = \left\{ v_* \in \mathbb{R}^N \text{ s.t. } \frac{v - v_*}{|v - v_*|} \cdot \eta \leq \mu \text{ and } |v_*| \leq (1 - \mu)^{-1} \right\}$$

and the truncation of the limit ρ ,

$$\rho_\mu(x, t, v, \eta) = \int \mathbb{1}_{V(v, \mu, \eta)}(v_*) G(v_*) dv_*.$$

This way, the uniform quasi-ellipticity estimate (9) implies that for every $\mu \in (0, 1)$ there exists a $\nu(\mu)$ such that

$$\bar{A}_{ij}^n(t, x, v) \eta_i \eta_j \geq \nu(\mu) \int \mathbb{1}_{V(v, \mu, \eta)}(v_*) G_*^n dv_*.$$

We then define the approximated quantum density and its V truncation:

$$\rho^n(x, t) = \int G_*^n dv_*, \quad \rho_\mu^n(x, t, v, \eta) = \int \mathbb{1}_{V(v, \mu, \eta)}(v_*) G_*^n dv_*.$$

Notice that ρ_μ and ρ_μ^n are continuous in (v, η) by dominated convergence.

First, as $\mu \nearrow 1$, we have $V(t, x, v, \eta) \nearrow \mathbb{R}^N \setminus (v + \mathbb{R}\eta)$, and thus by Beppo Levi's lemma,

$$\rho_\mu(t, x, v, \eta) \rightarrow \rho(t, x) \text{ a.e. } (t, x) \in (0, T) \times B_R, \forall (v, \eta) \in B_R \times S^{N-1},$$

and we upgrade the pointwise convergence in the (v, η) variables to a uniform convergence by Dini's theorem.

To obtain uniformity in (t, x) we have, by Egorov's theorem, that there exists a measurable set E_1 , with $|E_1| < \varepsilon/2$, such that the convergence above is uniform in $E_1^c \times B_R \times S^{N-1}$. Thus, we have that there exists a $\mu_0 = \mu_0(\varepsilon, \alpha)$ such that $\rho_\mu \geq \alpha/2$ in $(K_\alpha \cap E_1^c) \times B_R \times S^{N-1}$ for every $\mu \in [\mu_0, 1)$. We then fix such a μ .

Next, by the hypothesis over G^n , we have that

$$(12) \quad \rho_\mu^n(t, x, v, \eta) \rightarrow \rho_\mu(t, x, v, \eta) \text{ a.e. } (t, x) \in (0, T) \times B_R, \forall (v, \eta) \in B_R \times S^{N-1}.$$

We argue that the sequence ρ_μ^n is uniformly equicontinuous in (v, η) . Indeed, for some $\varepsilon' > 0$ we have, by uniform continuity of

$$\begin{cases} B_R \times S^{N-1} & \rightarrow L^\infty(\mathbb{R}_v^N) \\ (v, \eta) & \mapsto \mathbb{1}_{V_{v, \mu, \eta}} \end{cases}$$

that

$$|(v_2, \eta_2) - (v_1, \eta_1)| \leq \delta \implies \|\mathbb{1}_{V(v_2, \mu, \eta_2)} - \mathbb{1}_{V(v_1, \mu, \eta_1)}\|_{L^\infty} \leq \varepsilon'$$

for some $\delta > 0$, and therefore

$$\begin{aligned} |\rho_\mu^n(x, t, v_2, \eta_2) - \rho_\mu^n(x, t, v_1, \eta_1)| &\leq \int |\mathbb{1}_{V(v_2, \mu, \eta_2)} - \mathbb{1}_{V(v_1, \mu, \eta_1)}| g_*^n dv_* \\ &\leq \varepsilon' |B_{(1-\mu)^{-1}}|. \end{aligned}$$

This in turn implies that the convergence (12) is uniform in (v, η) . One more application of Egorov's theorem as before implies that there exists a measurable set E_2 with $|E_2| < \varepsilon/2$ and an $n_0 = n_0(\varepsilon, \alpha)$ such that

$$\rho_\mu^n \geq \frac{\alpha}{4} \text{ in } (K_\alpha \cap E^c) \times B_R \times S^{N-1}, \forall n \geq n_0$$

where $E = E_1 \cup E_2$ and $|E| < \varepsilon$. The result then follows. \square

3.3. Almost everywhere compactness. Intuitively, we can interpret the velocity averaging result, Lemma 1, as a "compactness in t, x ", provided by the transport operator. In this way, it is expected that if we can find some compactness result in the variable v , then we must have compactness for the sequence of functions as a whole.

In this section we show that this intuitive notion indeed holds. That is, that velocity averaging together with compactness in v actually imply compactness almost everywhere for the sequence of functions. Even more, we were able to alleviate the need for a "complete compactness" in v to achieve a more general, if more technical, result.

This extra generality, however, is necessary for our application since the diffusion matrix of the equation is not uniformly elliptic, as seen in the Section 3.2.

Proposition 2. *Let $(\Phi^n)_n$ be a sequence of functions such that $\Phi^n \xrightarrow{*} \Phi$ in $L^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$. Suppose that, for each $T, R > 0$,*

1) Quasi-bounded in v : For every $\varepsilon, \alpha > 0$, there exists a $C = C(\varepsilon, \alpha)$ and a measurable set E , with $|E| \leq \varepsilon$ such that

$$\iiint_{(K_\alpha \cap E^c) \times B_R} |\nabla_v \Phi^n|^2 dt dx dv \leq C(\varepsilon, \alpha)$$

where $K_\alpha = \{(t, x) \in [0, T] \times B_R : \int \Phi dv > \alpha\}$,

2) Velocity averages: for every $\varphi \in C_c^\infty(\mathbb{R}_v^N)$, $(\int \Phi^n \varphi dv)_n$ is compact in $L^1([0, T] \times B_R)$.

Thus, passing to a subsequence, we have $\Phi^n \rightarrow \Phi$ almost everywhere in $(0, \infty) \times \mathbb{R}_{x,v}^{2N}$.

Proof. For each $T, R > 0$ and $\varphi \in C_c^\infty(\mathbb{R}_v^N)$ we have, passing to a subsequence and using uniqueness of the weak limit, that

$$(13) \quad \int \Phi^n \varphi dv \rightarrow \int \Phi \varphi dv \text{ in } L^1([0, T] \times B_R).$$

So far, the subsequence we take depends on the function φ chosen, but we can construct a subsequence that works for all φ . Indeed, using that $C_c^\infty(\mathbb{R}_v^N)$ is dense in $L^1(\mathbb{R}_v^N)$ and a diagonal argument we may suppose that $\varphi \in L^1(\mathbb{R}_v^N)$. Next, taking a Schauder basis of $L^1(\mathbb{R}_v^N)$ and applying another diagonal argument, we may assume that the convergence (13) holds for every $\varphi \in L^1(\mathbb{R}_v^N)$.

Notice however that at this stage the subsequence we take still depends on the T and R chosen. We could, of course, get rid of this limitation by a further diagonal extraction argument, which will give us a sequence converging in $L^1_{loc}((0, \infty) \times \mathbb{R}_x^N)$, but this is not necessary at the moment for the argument, and we will save this approach for the last part of the proof, where we will also deal with the v variable.

Let $\eta \in C_c^\infty(\mathbb{R}_v^N)$ be a positive function with $\int \eta dv = 1$, supported in the unit ball of v and consider the mollifier $\eta_\delta(v) = \frac{1}{\delta^N} \eta\left(\frac{v}{\delta}\right)$. We will note $\Phi_\delta^n := \Phi^n *_v \eta_\delta$ and $\Phi_\delta := \Phi *_v \eta_\delta$. Pick an $R_0 > 0$ and consider $R_1 = R_0 + 1$. The velocity averaging result (13), applied to a $T > 0$ and $R = R_1$ implies we have, for every $\delta > 0$,

$$\Phi_\delta^n \rightarrow \Phi_\delta \text{ a.e. in } [0, T] \times B_{R_1} \times B_{R_1}$$

and thus by dominated convergence, $(\Phi_\delta^n)_n$ converges in $L^1((0, T) \times B_{R_1} \times B_{R_1})$.

Now, by the definition of Φ_δ^n , we have

$$\begin{aligned} \int_{B_{R_0}} |\Phi^n - \Phi_\delta^n| dv &= \int_{B_{R_0}} \left| \Phi^n(v) - \int_{B_\delta} \frac{1}{\delta^N} \eta\left(\frac{v_*}{\delta}\right) \Phi^n(v - v_*) dv_* \right| dv \\ &\leq \int_{B_{R_0}} \int_{B_\delta} \frac{1}{\delta^N} \eta\left(\frac{v_*}{\delta}\right) |\Phi^n(v - v_*) - \Phi^n(v)| dv_* dv \end{aligned}$$

By the mean value theorem, it follows that this is bounded by

$$\int_{B_{R_0}} \int_{B_\delta} \int_0^1 \frac{1}{\delta^N} \eta\left(\frac{v_*}{\delta}\right) |\nabla_v \Phi^n(v - \tau v_*)| |v_*| d\tau dv_* dv.$$

If δ is sufficiently small, then $v - \tau v_* \in B_{R_1}$, for every $v \in B_{R_0}$ and $\tau \in [0, 1]$. This way, the above integral is smaller than

$$\begin{aligned} \int_{B_{R_1}} \int_{B_\delta} \int_0^1 \frac{1}{\delta^N} \eta\left(\frac{v_*}{\delta}\right) |\nabla_v \Phi^n(v)| |v_*| d\tau dv_* dv &\leq \delta \int_{B_{R_1}} |\nabla_v \Phi^n(v)| \left(\int_{B_\delta} \frac{1}{\delta^N} \eta\left(\frac{v_*}{\delta}\right) dv_* \right) dv \\ &= \delta \int_{B_{R_1}} |\nabla_v \Phi^n(v)| \end{aligned}$$

Further integrating in $(t, x) \in K_\alpha \cap E^c$ we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \iiint_{(K_\alpha \cap E^c) \times B_{R_0}} |\Phi^n - \Phi_\delta^n| dt dx dv &\leq \delta \iiint_{(K_\alpha \cap E^c) \times B_{R_1}} |\nabla_v \Phi^n| dt dx dv \\ (14) \quad &\leq \delta (C(\varepsilon, \alpha))^{1/2} (|K_\alpha \cap E^c| |B_{R_1}|)^{1/2} \\ &\leq C_1(\varepsilon, \alpha) \delta, \end{aligned}$$

for $\delta > 0$ sufficiently small and $n \geq n_0(\alpha, \varepsilon)$.

From there, we can show that $(\Phi^n)_n$ is a Cauchy sequence in $L^1((0, T) \times B_{R_0} \times B_{R_0})$, because

$$\begin{aligned}
\iiint_{(0, T) \times B_{R_0} \times B_{R_0}} |\Phi^n - \Phi^m| dt dx dv &\leq \iiint_{(0, T) \times B_{R_0} \times B_{R_1}} |\Phi^n - \Phi^m| dt dx dv \\
&= \iint_{K_\alpha^c} \int_{B_{R_0}} |\Phi^n - \Phi^m| dt dx dv + \iint_E \int_{B_{R_0}} |\Phi^n - \Phi^m| dt dx dv \\
&\quad + \iint_{E^c \cap K_\alpha} \int_{B_{R_0}} |\Phi^n - \Phi_\delta^n| + |\Phi^m - \Phi_\delta^m| dt dx dv \\
&\quad + \iint_{E^c \cap K_\alpha} \int_{B_{R_0}} |\Phi_\delta^n - \Phi_\delta^m| dt dx dv \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For I_1 , Velocity averaging implies that $\int_{B_{R_0}} \Phi^n \rightarrow \int_{B_{R_0}} \Phi$, thus

$$\begin{aligned}
\limsup_{n, m \rightarrow \infty} I_1 &\leq \limsup_{n, m \rightarrow \infty} \iint_{K_\alpha^c} \int_{B_{R_0}} \Phi^n + \Phi^m dt dx dv \\
&= 2 \iint_{K_\alpha^c} \int_{B_{R_0}} \Phi dv dt dx \leq 2\alpha T |B_{R_1}|.
\end{aligned}$$

For I_2 , we use that $|\Phi^n| \leq M$ to show it's bounded by $2M|B_{R_0}||E| \leq 2M|B_{R_0}|\varepsilon$. We bound I_3 with (14) and lastly

$$\limsup_{n, m \rightarrow \infty} I_4 \leq \limsup_{n, m \rightarrow \infty} \|\Phi_\delta^n - \Phi_\delta^m\|_{L^1((0, T) \times B_{R_1} \times B_{R_1})} = 0,$$

since $(\Phi_\delta^n)_n$ is a Cauchy sequence.

Finally, we have shown that

$$\limsup_{n, m \rightarrow \infty} \int_{(0, T) \times B_{R_0} \times B_{R_0}} |\Phi^n - \Phi^m| dt dx dv \leq 2\alpha T |B_{R_1}| + 2M|B_{R_0}|\varepsilon + 2C_1(\alpha, \varepsilon)\delta$$

and taking $\delta \rightarrow 0$, then $\alpha, \varepsilon \rightarrow 0$, we find that $\Phi^n \rightarrow \Phi$ in $L^1((0, T) \times B_{R_0} \times B_{R_0})$.

Since the $R_0 > 0$ and the $T > 0$ are arbitrary, we can do a diagonal extraction to arrive at a subsequence such that $\Phi^n \rightarrow \Phi$ in $L^1_{loc}((0, \infty) \times \mathbb{R}^{2N}_{x,v})$, which passing to a further subsequence if necessary, implies convergence almost everywhere. \square

Now we bring everything together for a proof of the main proposition of this section.

Proof of Proposition 1. The integral bounds obeyed by the sequence g^n , together with the fact that $0 \leq g^n \leq 1$ for every n allows us, using Dunford-Pettis, to extract a subsequence converging weakly in $L^1_{loc}((0, \infty); L^1(\mathbb{R}^{2N}_{x,v}))$. By Scheffé's lemma, to show the full convergence in L^1 , we just need to show a.e. convergence.

Lemma 1 gives us directly that the velocity averages are compact. On the other hand inequality (10), together with Lemma 2, gives us that for each $R, \varepsilon, \alpha > 0$ there exists $C, n_0 > 0$ and a measurable set E , with $|E| \leq \varepsilon$, such that

$$\iiint_{(K_\alpha \cap E^c) \times B_R} |\nabla_v g^n|^2 dt dx dv \leq C,$$

for every $n \geq n_0$. Proposition 2 then applies and we can further extract an a.e. convergent sequence, thus proving the result. \square

4. CAUCHY THEORY FOR THE APPROXIMATED EQUATION

In this section, we will define an equation that approximates LFD and construct solutions to it. The main goal here is to construct an approximation scheme that satisfies the hypothesis from Proposition 1 and then pass to the limit to recover LFD, which we will do in Section 5.

We define this approximated equation by considering a smooth collision kernel a , smooth initial data f_0 and by adding a vanishing viscosity term $\varepsilon \Delta_v f$, which will guarantee some regularity for the solutions of this equation.

However, we still need our approximate collision kernel to respect the conservation laws, and for this we need it to satisfy

$$(15) \quad a(z) \geq 0, \quad a(-z) = a(z) \quad \text{and} \quad a(z)z \cdot z = 0,$$

for every $z \in \mathbb{R}^N$.

Our main objective, for this section, will be to show the following result:

Proposition 3. *Let $a \in \mathcal{S}(\mathbb{R}^N)$ be a collision kernel satisfying (15) and $f_0 \in \mathcal{S}(\mathbb{R}_{x,v}^{2N})$ an initial data such that, for some $C'_1, C'_2, \alpha_0 > 0$,*

$$C'_1 e^{-\alpha_0(|x|^2 + |v|^2)} \leq f_0(x, v) \leq \frac{C'_2 e^{-\alpha_0(|x|^2 + |v|^2)}}{1 + C'_2 e^{-\alpha_0(|x|^2 + |v|^2)}}.$$

Then, there exists a solution $f \in C((0, \infty); \mathcal{D}'(\mathbb{R}_{x,v}^{2N})) \cap L^2((0, \infty) \times \mathbb{R}_x^N; H^1(\mathbb{R}_v^N))$ to the approximated problem

$$(16) \quad \begin{cases} \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} = \frac{\partial}{\partial v_j} \left\{ \bar{a}_{ij}^f \frac{\partial f}{\partial v_i} - \bar{b}_j^f f(1-f) \right\} + \varepsilon \Delta_v f \\ f|_{t=0} = f_0. \end{cases}$$

*in the sense of Definition 1, where $\bar{a}_{ij}^f \equiv a *_{v}(f(1-f))$ and $\bar{b}_i^f \equiv \frac{\partial a_{ij}}{\partial v_j} *_{v} f$.*

Moreover, for each $T > 0$ there exists a $C > 0$ such that

$$\iiint_{(0,T) \times \mathbb{R}_{x,v}^{2N}} e^{\alpha(|x|^2 + |v|^2)} |\nabla_v f|^2 dx dv dt \leq C$$

and there exist $C_1, C_2, \alpha > 0$ such that

$$C_1 e^{-\alpha(|x|^2 + |v|^2)} \leq f(t, x, v) \leq \frac{C_2 e^{-\alpha(|x|^2 + |v|^2)}}{1 + C_2 e^{-\alpha(|x|^2 + |v|^2)}}$$

for a.e. $(t, x, v) \in (0, T) \times \mathbb{R}_{x,v}^{2N}$.

Our approach will be to construct an iterative scheme in a linear equation. Given $f_0, g \in C^\infty((0, T) \times \mathbb{R}_{x,v}^{2N})$, consider the linear approximated problem

$$(17) \quad \begin{cases} \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} = \frac{\partial}{\partial v_j} \left\{ \bar{a}_{ij}^g \frac{\partial f}{\partial v_i} \right\} - \bar{b}_j^g (1-2g) \frac{\partial f}{\partial v_j} - \frac{\partial \bar{b}_j^g}{\partial v_j} g(1-f) + \varepsilon \Delta_v f \\ f|_{t=0} = f_0. \end{cases}$$

We construct an approximation scheme to equation (16) by induction, taking f^{n+1} as the solution of (17) when we let $g = f^n$. Then, equation (16) will come as a limit of this scheme when $n \rightarrow \infty$. This is the core idea to the existence part of Proposition 3, and we will see that in more detail when we tackle its proof. For now, let us focus on showing some properties of equation (17) that are independent of g (which in the approximating scheme will imply that they are uniform in n).

It has long been observed that although the diffusion acts only in the v variable, the solutions to this type of problem are regular in both variables. This behavior, called hypoellipticity, was first

identified by Kolmogorov [13] for the equation $\partial_t f + v \cdot \nabla_x f - \Delta_v f = 0$, while studying an stochastic process, which he was then able to solve using a fundamental solution.

The pursuit for fundamental solutions then expanded to a wider class of equations, such as

$$(18) \quad \mathcal{L}(f) = -\frac{\partial f}{\partial t} - v_i \frac{\partial f}{\partial x_i} + a_{ij} \frac{\partial^2 f}{\partial v_i \partial v_j} + a_i \frac{\partial f}{\partial v_i} + af = 0,$$

with $(a_{ij})_{ij}$ uniformly elliptic, which were extensively studied in 1951 by Weber [16], then in 1964 by Il'in [12], who constructed fundamental solutions for these equations that imply the existence of classical solutions. In Hormander's work, we find a more precise definition of hypoellipticity, in which the operator \mathcal{L} is called hypoelliptic if f is regular wherever $\mathcal{L}(f)$ is regular. It was even shown in [11] that equations like (18) with C^∞ coefficients are in fact hypoelliptic. Thus, once we construct a solution (even a weak one) to the equation $\mathcal{L}(f) = 0$ in $(0, T) \times \mathbb{R}_{x,v}^{2N}$, it will automatically be $C^\infty((0, T) \times \mathbb{R}_{x,v}^{2N})$.

Thus, if $g \in C^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ and $f_0 \in C^\infty(\mathbb{R}_{x,v}^{2N})$, we know there exists a solution $f \in C^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ to (17), where $f(t) \rightarrow f_0$ when $t \rightarrow 0^+$. Once we have our smooth solution, by a comparison principle, we can show that if $0 \leq g \leq 1$, then $0 \leq f \leq 1$, and thus the linear approximated equation preserves the Pauli exclusion principle. In particular, the matrix $(\bar{a}_{ij}^g)_{ij}$ is non-negative.

With a little more effort, however, we can use a comparison principle to prove a Gaussian decay for the f function, independent of g , as is shown in the next lemma

Lemma 3. *Let $0 \leq g \leq 1$ be a smooth function and suppose the initial data f_0 satisfies*

$$C_1' e^{-\alpha_0(|x|^2 + |v|^2)} \leq f_0(x, v) \leq \frac{C_2' e^{-\alpha_0(|x|^2 + |v|^2)}}{1 + C_2' e^{-\alpha_0(|x|^2 + |v|^2)}}$$

Then, for every $t > 0$ there exists constants $\alpha, C, C_1, C_2 > 0$, independent of g , such that the solution f to (17) satisfies

$$(19) \quad C_1 e^{-\alpha(|x|^2 + |v|^2)} \leq f(t, x, v) \leq \frac{C_2 e^{-\alpha(|x|^2 + |v|^2)}}{1 + C_2 e^{-\alpha(|x|^2 + |v|^2)}},$$

for $t \in [0, T]$.

Proof. We start by showing the inequality on the right side. Define the differential operator

$$\mathcal{L}(f) = \frac{\partial}{\partial v_j} \left\{ \bar{a}_{ij} \frac{\partial f}{\partial v_i} \right\} - \bar{b}_j (1 - 2g) \frac{\partial f}{\partial v_j} - \frac{\partial \bar{b}_j}{\partial v_j} g (1 - f) + \varepsilon \Delta_v f - \frac{\partial f}{\partial t} - v_i \frac{\partial f}{\partial x_i},$$

where, for brevity, we have dropped the reference to g in \bar{a} and \bar{b} .

Let $\varphi \in C^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$. We have that

$$\begin{aligned} \mathcal{L} \left(\frac{\exp \varphi}{1 + \exp \varphi} \right) &= \exp(-\varphi) \left[\varepsilon \frac{\partial^2 \varphi}{\partial v_i^2} - \varepsilon \left(\frac{\partial \varphi}{\partial v_i} \right)^2 + \frac{\partial \bar{a}_{ij}}{\partial v_i} \frac{\partial \varphi}{\partial v_j} + \bar{a}_{ij} \frac{\partial^2 \varphi}{\partial v_i \partial v_j} - \bar{a}_{ij} \frac{\partial \varphi}{\partial v_i} \frac{\partial \varphi}{\partial v_j} \right. \\ &\quad \left. - \bar{b}_j (1 - 2g) \frac{\partial \varphi}{\partial v_j} - v_i \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial t} \right] + \frac{\partial \bar{b}_j}{\partial v_j} g \frac{\exp \varphi}{1 + \exp \varphi} \end{aligned}$$

and thus, since \bar{a} is a non-negative matrix,

$$\begin{aligned} \mathcal{L} \left(\frac{\exp \varphi}{1 + \exp \varphi} \right) - \mathcal{L}(f) &\leq \exp(-\varphi) \left[\varepsilon \frac{\partial^2 \varphi}{\partial v_i^2} - \varepsilon \left(\frac{\partial \varphi}{\partial v_i} \right)^2 + \frac{\partial \bar{a}_{ij}}{\partial v_i} \frac{\partial \varphi}{\partial v_j} + \bar{a}_{ij} \frac{\partial^2 \varphi}{\partial v_i \partial v_j} \right. \\ &\quad \left. - \bar{b}_j (1 - 2g) \frac{\partial \varphi}{\partial v_j} - \frac{\partial \bar{b}_j}{\partial v_j} g - v_i \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial t} \right]. \end{aligned}$$

Then choosing $\varphi = k_0 + k_1 t - k_2 \frac{|x|^2 + |v|^2}{1+t}$, it follows, using Young's inequality,

$$\begin{aligned} \mathcal{L} \left(\frac{\exp \varphi}{1 + \exp \varphi} \right) - \mathcal{L}(f) &\leq \exp(-\varphi) \left[-\varepsilon N \frac{2k_2}{1+t} - \varepsilon \left(\frac{2k_2}{1+t} \right)^2 |v|^2 + (\|\nabla_v \bar{a}\|_\infty + \|\bar{b}\|_\infty) \frac{2k_2}{1+t} |v| \right. \\ &\quad \left. + \|\nabla_v \bar{b}\|_\infty + \frac{2k_2}{1+t} |x||v| - k_1 - k_2 \frac{|x|^2 + |v|^2}{(1+t)^2} \right] \\ &\leq \exp(-\varphi) \left[\frac{1}{2\varepsilon} (\|\nabla_v \bar{a}\|_\infty + \|\bar{b}\|_\infty)^2 + \|\nabla_v \bar{b}\|_\infty + \frac{1}{2\varepsilon} |x|^2 - k_1 - k_2 \frac{|x|^2}{(1+t)^2} \right]. \end{aligned}$$

Therefore, choosing $k_1 \geq \frac{1}{2\varepsilon} (\|\nabla_v \bar{a}\|_\infty + \|\bar{b}\|_\infty)^2 + \|\nabla_v \bar{b}\|_\infty$ and $k_2 \geq \frac{(1+T)^2}{2\varepsilon}$, the above quantity is negative, for $t \in [0, T]$.

Finally, choosing $k_0 = \log C'_2$ and further taking $k_2 \geq \alpha_0$, we have that $f_0 \leq \frac{\exp \varphi_0}{1 + \exp \varphi_0}$ and the comparison principle implies, for $t \in [0, T]$,

$$f \leq \frac{C'_2 e^{k_1 t} e^{-\frac{k_2}{1+t}(|x|^2 + |v|^2)}}{1 + C_2 e^{k_1 t} e^{-\frac{k_2}{1+t}(|x|^2 + |v|^2)}} \leq \frac{C'_2 e^{k_1 T} e^{-\frac{1+T}{2\varepsilon}(|x|^2 + |v|^2)}}{1 + C'_2 e^{k_1 T} e^{-\frac{1+T}{2\varepsilon}(|x|^2 + |v|^2)}}.$$

Letting $C_2 = C'_2 e^{k_1 T}$ and $\alpha = \frac{1}{\varepsilon(1+T)}$ the first inequality follows. We now turn to prove the left inequality. Notice that

$$\begin{aligned} \mathcal{L}(\exp \varphi) &= \exp \varphi \left[\varepsilon \frac{\partial^2 \varphi}{\partial v_i^2} + \varepsilon \left(\frac{\partial \varphi}{\partial v_i} \right)^2 + \frac{\partial \bar{a}_{ij}}{\partial v_i} \frac{\partial \varphi}{\partial v_j} + \bar{a}_{ij} \frac{\partial^2 \varphi}{\partial v_i \partial v_j} + \bar{a}_{ij} \frac{\partial \varphi}{\partial v_i} \frac{\partial \varphi}{\partial v_j} \right. \\ &\quad \left. - \bar{b}_j (1 - 2g) \frac{\partial \varphi}{\partial v_j} + \frac{\partial \bar{b}_j}{\partial v_j} g - v_i \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial t} \right]. \end{aligned}$$

Choosing $\varphi = k_0 - k_1 t - k_2(1+t)(|x|^2 + |v|^2)$, it follows, using Young's inequality,

$$\begin{aligned} \mathcal{L}(\exp \varphi) &\geq \exp \varphi \left[-\varepsilon 2k_2(1+t)N + \varepsilon 4k_2^2(1+t)^2 |v|^2 - \frac{\partial \bar{a}_{ij}}{\partial v_i} k_2(1+t) 2v_j - \bar{a}_{ij} 2k_2(1+t) \delta_{ij} \right. \\ &\quad \left. + \bar{b}_j (1 - 2g) k_2(1+t) 2v_j + \frac{\partial \bar{b}_j}{\partial v_j} g + k_2(1+t) 2v_i x_i + k_1 + k_2(|x|^2 + |v|^2) \right] \\ &\geq \exp \varphi \left[-\varepsilon 2k_2(1+t)N - \frac{1}{2\varepsilon} (\|\nabla_v \bar{a}\|_\infty + \|\bar{b}\|_\infty)^2 - \|\bar{a}\|_\infty 2k_2(1+t) \right. \\ &\quad \left. - \|\bar{b}\|_\infty - \frac{1}{2\varepsilon} |x|^2 + k_1 + k_2(|x|^2 + |v|^2) \right]. \end{aligned}$$

Thus, choosing $k_2 \geq \frac{1}{2\varepsilon}$ and k_1 sufficiently large, we have $\mathcal{L}(\exp \varphi) \geq 0 \geq \mathcal{L}(f)$. Further, we can suppose $k_2 \geq \alpha_0$ and choose $k_0 = \log C'_2$, which ensures $f_0 \geq \exp \varphi_0$ and thus the comparison principle implies that, for $t \in [0, T]$,

$$f \geq C'_1 e^{-k_1 t} e^{-k_2(1+t)(|x|^2 + |v|^2)} \geq C'_1 e^{-k_1 T} e^{-\frac{1+T}{2\varepsilon}(|x|^2 + |v|^2)}$$

and the left inequality follows by letting $C_1 = C'_1 e^{-k_1 T}$.

Note that the constants k_0 , k_1 and k_2 we choose for each of the inequalities depend only on ε and the L^∞ norms of \bar{a} , \bar{b} and their derivatives with respect to v , and thus these constants are uniform with respect to g . \square

We will also need decay estimates on the derivatives of f , uniform in g . However, these estimates are very difficult to obtain by means of a comparison principle. Here, we use energy methods to deduce a uniform estimate with weights, which will be enough for what we need.

Lemma 4. Suppose f is a solution of (17) such that f and g both satisfy inequality (19), with the same constants. Then, for every $T > 0$ there exists a $C > 0$, independent of g , such that

$$(20) \quad \iiint_{(0,T) \times \mathbb{R}_{x,v}^{2N}} e^{\alpha(|x|^2+|v|^2)} |\nabla_v f|^2 dx dv dt \leq C,$$

where this α is the same one as in Lemma 3.

Proof. Since we don't have decay estimates for the derivatives of f , in order to justify the following integrals, let $\psi^n = \psi^n(x, v)$ be such that $\psi^n = \psi(\cdot/n)$, where $\psi \in C_c^\infty(\mathbb{R}_{x,v}^{2N})$, $\psi(0) = 1$ and $\iint \psi dx dv = 1$.

For brevity, we will define $\bar{A}_{ij} = \bar{a}_{ij} + \varepsilon \delta_{ij}$. Multiplying (17) by $e^{\alpha(|x|^2+|v|^2)} f \psi$ and integrating in x, v , we have,

$$(21) \quad \frac{1}{2} \frac{d}{dt} \int_{xv} e^{\alpha(|x|^2+|v|^2)} f^2 \psi^n + \int_{xv} v_i \frac{\partial f}{\partial x_i} e^{\alpha(|x|^2+|v|^2)} f \psi^n = \int_{xv} \frac{\partial}{\partial v_j} \left\{ \bar{A}_{ij} \frac{\partial f}{\partial v_i} \right\} e^{\alpha(|x|^2+|v|^2)} f \psi^n \\ - \int_{xv} \bar{b}_j (1 - 2g) \frac{\partial f}{\partial v_j} e^{\alpha(|x|^2+|v|^2)} f \psi^n - \int_{xv} \frac{\partial \bar{b}_j}{\partial v_j} g (1 - f) e^{\alpha(|x|^2+|v|^2)} f \psi^n$$

For the first integral on the right-hand side of (21) we have, integrating by parts in v ,

$$\int_{xv} \frac{\partial}{\partial v_j} \left\{ \bar{A}_{ij} \frac{\partial f}{\partial v_i} \right\} e^{\alpha(|x|^2+|v|^2)} f \psi^n = -2\alpha \int_{xv} \bar{A}_{ij} \frac{\partial f}{\partial v_i} f v_j e^{\alpha(|x|^2+|v|^2)} \psi^n \\ - \int_{xv} \bar{A}_{ij} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} e^{\alpha(|x|^2+|v|^2)} \psi^n - \int_{xv} \bar{A}_{ij} \frac{\partial f}{\partial v_i} f e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi^n}{\partial v_i}.$$

Using the ellipticity of the matrix \bar{A} , the first two integrals are bounded by

$$2\alpha \int_{xv} |\bar{A}_{ij}| \left| \frac{\partial f}{\partial v_i} \right| |f| |v_j| e^{\alpha(|x|^2+|v|^2)} \psi^n - \varepsilon \int_{xv} |\nabla_v f|^2 e^{\alpha(|x|^2+|v|^2)} \psi^n$$

and by Young's inequality, this is bounded by

$$\frac{2\alpha^2 N^2}{\varepsilon} \|\bar{A}\|_\infty^2 \int_{xv} |f|^2 |v|^2 e^{\alpha(|x|^2+|v|^2)} \psi^n - \frac{\varepsilon}{2} \int_{xv} |\nabla_v f|^2 e^{\alpha(|x|^2+|v|^2)} \psi^n.$$

Again using Young's inequality, we have

$$\int_{xv} |\bar{b}_j| |1 - 2g| |f| \left| \frac{\partial f}{\partial v_j} \right| e^{\alpha(|x|^2+|v|^2)} \psi^n \leq \frac{\varepsilon}{4} \int_{xv} |\nabla_v f|^2 e^{\alpha(|x|^2+|v|^2)} \psi^n + \frac{N}{\varepsilon} \|\bar{b}\|_\infty^2 \int_{xv} |f|^2 e^{\alpha(|x|^2+|v|^2)} \psi^n.$$

and since $0 \leq f \leq 1$,

$$\int_{xv} \left| \frac{\partial \bar{b}_j}{\partial v_j} \right| |g| |1 - f| |f| e^{\alpha(|x|^2+|v|^2)} \psi^n \leq N \|\nabla_v \bar{b}\|_\infty \int_{xv} |g| |f| e^{\alpha(|x|^2+|v|^2)} \psi^n$$

Finally, integration by parts, we get

$$\int_{xv} v_i \frac{\partial f}{\partial x_i} e^{\alpha(|x|^2+|v|^2)} f \psi^n = -\alpha \int_{xv} f^2 x_i v_i e^{\alpha(|x|^2+|v|^2)} \psi^n - \int_{xv} \frac{f^2}{2} v_i e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi^n}{\partial v_i}$$

and thus that there exists a $C > 0$ such that

$$\frac{d}{dt} \int_{xv} \frac{f^2}{2} e^{\alpha(|x|^2+|v|^2)} \psi^n \leq -\frac{\varepsilon}{4} \int_{xv} |\nabla_v f|^2 e^{\alpha(|x|^2+|v|^2)} \psi^n \\ + C \int_{xv} ((|v|^2 + |x||v| + 1) f^2 + |g||f|) e^{\alpha(|x|^2+|v|^2)} \psi^n + R^n$$

where the remainder R^n is given by

$$R^n = - \int_{xv} \bar{A}_{ij} \frac{\partial f}{\partial v_i} f e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi^n}{\partial v_i} + \int_{xv} \frac{f^2}{2} v_i e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi^n}{\partial v_i}.$$

Integrating over $t \in (0, T)$, we have that

$$(22) \quad \int_{txv} |\nabla_v f|^2 e^{\alpha(|x|^2+|v|^2)} \psi^n \leq \frac{2}{\varepsilon} \int_{txv} (f(0)^2 - f(T)^2) e^{\alpha(|x|^2+|v|^2)} \psi^n \\ + \frac{4C}{\varepsilon} \int_{txv} ((|v|^2 + |x||v| + 1)f^2 + |g||f|) e^{\alpha(|x|^2+|v|^2)} \psi^n + \frac{4}{\varepsilon} \int_0^T R^n,$$

where we have used \int_{txv} to denote $\int_0^T \int_{xv} dt$.

Now we release the truncation. The bound (19), satisfied by f and g , imply that the first two integrals are uniformly bounded in n . Let us show that the remainder $\int R^n$ vanishes in the limit $n \rightarrow \infty$. Indeed, integrating by parts and using the definition of ψ^n ,

$$\int_{txv} \bar{A}_{ij} \frac{\partial}{\partial v_i} \left(\frac{f^2}{2} \right) e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi^n}{\partial v_i} = -\frac{1}{n} \int_{(0,T) \times B_n \times B_n} \frac{\partial \bar{A}_{ij}}{\partial v_i} \frac{f^2}{2} e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi}{\partial v_j} \left(\frac{\cdot}{n} \right) \\ - \frac{\alpha}{n} \int_{(0,T) \times B_n \times B_n} \bar{A}_{ij} f^2 v_i e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi}{\partial v_j} \left(\frac{\cdot}{n} \right) \\ - \frac{1}{n^2} \int_{(0,T) \times B_n \times B_n} \bar{A}_{ij} \frac{f^2}{2} e^{\alpha(|x|^2+|v|^2)} \frac{\partial^2 \psi}{\partial v_i \partial v_j} \left(\frac{\cdot}{n} \right).$$

These integrals are uniformly bounded in n from the estimates (19), and thus this quantity converges to zero. Similarly,

$$\int_{txv} \frac{f^2}{2} v_i e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi^n}{\partial v_i} = \frac{1}{n} \int_{(0,T) \times B_n \times B_n} \frac{f^2}{2} v_i e^{\alpha(|x|^2+|v|^2)} \frac{\partial \psi}{\partial v_i} \left(\frac{\cdot}{n} \right)$$

also converges to zero. Thus, we can pass to the limit in (22), leading $|\nabla_v f|^2 e^{\alpha(|x|^2+|v|^2)} \in L^1((0, T) \times \mathbb{R}_{x,v}^{2N})$. Notice this integral is independent of the function g , as long as it obeys the bound (19). \square

Now, we can use the bounds shown above to build an iterative process, which we will show converges to a solution of the approximate problem (16).

Proof of Proposition 3. Throughout this proof, we will notate, for every $p \in [1, \infty]$, L^p the space $L_{loc}^p((0, \infty); L^p(\mathbb{R}_{x,v}^{2N}))$, L_{loc}^p the space $L_{loc}^p((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ and $L_{tx}^2 H_v^1$ the space $L_{loc}^2((0, \infty); L^2(\mathbb{R}_x^N; H^1(\mathbb{R}_v^N)))$. Notice all of these spaces are local in time.

Part 1: The iterative process

The existence for the nonlinear regularized equation is obtained through an iterative process. Let f^0 be the solution of the Kolmogorov equation $\partial_t f + v \cdot \nabla_x f - \varepsilon \Delta_v f = 0$, with initial data f_0 . For $k \geq 1$, let f^k be the solution of problem (17) with $g = f^{k-1}$, that is, solution to the equation with initial data f_0 , where we note \bar{a}^{k-1} and \bar{b}^{k-1} instead of $\bar{a}^{f^{k-1}}$ and $\bar{b}^{f^{k-1}}$, respectively. That is, we consider the iterative process

$$(23) \quad \begin{cases} \frac{\partial f^k}{\partial t} + v_i \frac{\partial f^k}{\partial x_i} = \frac{\partial}{\partial v_j} \left\{ \bar{a}_{ij}^{k-1} \frac{\partial f^k}{\partial v_i} \right\} - \bar{b}_j^{k-1} (1 - 2f^{k-1}) \frac{\partial f^k}{\partial v_j} - \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1 - f^k) + \varepsilon \Delta_v f^k \\ f^k|_{t=0} = f_0. \end{cases}$$

As mentioned at the beginning of the section we know that for every $k \in \mathbb{N}$, this problem has a solution $f^k \in C^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$. We can extend this function to $C([0, \infty) \times \mathbb{R}_{x,v}^{2N})$ by continuity and this will imply $f^k(0) = f_0$.

Note that the bounds (19) and (20) found earlier, being independent of g , in this process become uniform in k . That is, for each $T > 0$ we have

$$(24) \quad C_1 e^{-\alpha(|x|^2 + |v|^2)} \leq f^k(t, x, v) \leq \frac{C_2 e^{-\alpha(|x|^2 + |v|^2)}}{1 + C_2 e^{-\alpha(|x|^2 + |v|^2)}}$$

and

$$(25) \quad \iiint_{(0,T) \times \mathbb{R}_{x,v}^{2N}} e^{\alpha(|x|^2 + |v|^2)} |\nabla_v f^k|^2 dx dv dt \leq C,$$

for constants $C, C_1, C_2, \alpha > 0$ depending on T but not on k .

We will then use the same techniques as in Section 2 to achieve compactness for this sequence of functions, i.e. we will use Proposition 2 to extract an a.e. convergent sequence. In this case, the extra regularity plays in our favor and this result is easily applied. Indeed, inequality (25) already shows us that the derivatives in v are bounded, and so we only have to show the compactness of the velocity averages.

Once again we will use Theorem 2, rewriting equation (23) as

$$\frac{\partial f^k}{\partial t} + v_i \frac{\partial f^k}{\partial x_i} = \frac{\partial H_i}{\partial v_i} - H.$$

Using that $0 \leq f^k \leq 1$, we have that

$$\begin{aligned} |H_i| &= \left| (\bar{a}_{ij}^{k-1} + \varepsilon \delta_{ij}) \frac{\partial f^k}{\partial v_i} \right| \leq (1 + \|a_{ij}\|_{L^1}) \left| \frac{\partial f^k}{\partial v_i} \right| \\ |H| &= \left| \bar{b}_j^{k-1} (1 - 2f^{k-1}) \frac{\partial f^k}{\partial v_j} + \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1 - f^k) \right| \leq \left\| \frac{\partial a_{ij}}{\partial v_i} \right\|_{L^1} \left| \frac{\partial f^k}{\partial v_j} \right| + \left\| \frac{\partial^2 a_{ij}}{\partial v_i \partial v_j} \right\|_{L^1} \end{aligned}$$

and thus these functions are uniformly bounded in L_{loc}^1 , which implies the velocity averages are compact.

Thus, Proposition 2 applies and there exists a subsequence, noted f^{k_l} , such that $f^{k_l} \xrightarrow{l \rightarrow \infty} f$ almost everywhere in $(0, \infty) \times \mathbb{R}_{x,v}^{2N}$. Moreover, the bound from inequality (24) implies this convergence holds in L^1 by dominated convergence. This, however, is not sufficient to pass the equation to the limit $l \rightarrow \infty$, as the coefficients of the equation in f^{k_l} depend on f^{k_l-1} , which we do not know to be compact.

The solution to this non-synchrony problem is then to consider the equation for f^{k_l+1} . The same argument as before gives us that this sequence of functions is compact, and therefore passing once more to a subsequence, with indices in m , we have that f^{k_m+1} is a convergent sequence. Further, since f^{k_m} is a subsequence of f^{k_l} , it is also convergent.

Since this sequence is uniformly bounded, it follows by interpolation that $f^{k_m} \rightarrow f$ in L^p , for every $p < \infty$. Also, by the uniform estimate (25), we may suppose, using Banach-Alaoglu, that this sequence converges weakly in $L_{tx}^2 H_v^1$.

Relabeling this last sequence simply as f^k we have, to summarize,

$$\begin{aligned} f^k &\rightarrow f \text{ in } L_{loc}^p((0, \infty); L^p(\mathbb{R}_{x,v}^{2N})), \text{ for every } p < \infty \\ f^k &\rightharpoonup f \text{ in } L_{loc}^2((0, \infty); L^2(\mathbb{R}_x^N; H^1(\mathbb{R}_v^N))), \end{aligned}$$

and we may as well suppose (possibly passing to a subsequence) that

$$f^k(t) \rightarrow f(t) \text{ in } L^1(\mathbb{R}_{x,v}^{2N}), \text{ for a.e. } t > 0.$$

Part 2: Passing the equation to the limit

We integrate this equation by parts against a test function $\varphi \in \mathcal{D}([0, \infty) \times \mathbb{R}_{x,v}^{2N})$ to obtain that, for every $t > 0$,

$$(26) \quad \begin{aligned} \int_{xv} f^k(t) \varphi(t) - \int_{xv} f_0 \varphi(0) - \int_0^t \int_{xv} f^k \frac{\partial \varphi}{\partial t} - \int_0^t \int_{xv} f^k v_i \frac{\partial \varphi}{\partial x_i} = \\ \int_0^t \int_{xv} (\bar{a}_{ij}^{k-1} + \varepsilon \delta_{ij}) f^k \frac{\partial^2 \varphi}{\partial v_i \partial v_j} + \int_0^t \int_{xv} \left(\frac{\partial \bar{a}_{ij}^{k-1}}{\partial v_i} + \bar{b}_j^{k-1} (1 - 2f^{k-1}) \right) f^k \frac{\partial \varphi}{\partial v_j} \\ + \int_0^t \int_{xv} \left[\left(\frac{\partial \bar{b}_j^{k-1}}{\partial v_j} (1 - 2f^{k-1}) - 2\bar{b}_j^{k-1} \frac{\partial f^{k-1}}{\partial v_j} \right) f^k - \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1 - f^k) \right] \varphi. \end{aligned}$$

Now, we want to pass equation (26) to the limit. We have, from the convergence of $(f^k)_k$ that the left hand side of this equation converges for almost every $t > 0$ to

$$\int_{xv} f(t) \varphi(t) - \int_{xv} f_0 \varphi(0) - \int_0^t \int_{xv} f \frac{\partial \varphi}{\partial t} - \int_0^t \int_{xv} f v_i \frac{\partial \varphi}{\partial x_i}.$$

For the right hand side, define $\bar{a}_{ij} \equiv a_{ij} *_{\nu} (f(1-f))$ and $\bar{b}_i \equiv \frac{\partial a_{ij}}{\partial v_j} *_{\nu} f$. From Young's inequality for convolutions, we deduce from the convergence of $(f^k)_k$ that $\bar{a}_{ij}^k \rightarrow \bar{a}_{ij}$, $\bar{b}_j^k \rightarrow \bar{b}_j$, $\frac{\partial \bar{a}_{ij}^k}{\partial v_j} \rightarrow \frac{\partial \bar{a}_{ij}}{\partial v_j}$ and $\frac{\partial \bar{b}_j^k}{\partial v_j} \rightarrow \frac{\partial \bar{b}_j}{\partial v_j}$ in L^p , for every $p < \infty$. This in turn implies that

$$\begin{aligned} (\bar{a}_{ij}^{k-1} + \varepsilon \delta_{ij}) f^k &\rightarrow (\bar{a}_{ij} + \varepsilon \delta_{ij}) f \\ \left(\frac{\partial \bar{a}_{ij}^{k-1}}{\partial v_i} + \bar{b}_j^{k-1} (1 - 2f^{k-1}) \right) f^k &\rightarrow \left(\frac{\partial \bar{a}_{ij}}{\partial v_i} + \bar{b}_j (1 - 2f) \right) f \\ \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} (1 - 2f^{k-1}) f^k &\rightarrow \frac{\partial \bar{b}_j}{\partial v_j} (1 - 2f) f \\ \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1 - f^k) &\rightarrow \frac{\partial \bar{b}_j}{\partial v_j} f (1 - f) \end{aligned}$$

in L^1 . For the last convergence, we note that $\bar{b}_j^{k-1} f^k \rightarrow \bar{b}_j f$ and $\frac{\partial f^k}{\partial v_j} \rightharpoonup \frac{\partial f}{\partial v_j}$ in L^2 , thus by a weak-strong result, we have that

$$\bar{b}_j^{k-1} f^k \frac{\partial f^k}{\partial v_j} \rightharpoonup \bar{b}_j f \frac{\partial f}{\partial v_j} \text{ in } L^1.$$

Thus, taking the limit $k \rightarrow \infty$ of (26), leads

$$(27) \quad \int_{xv} f(t)\varphi(t) - \int_{xv} f_0\varphi(0) - \int_0^t \int_{xv} f \frac{\partial \varphi}{\partial t} - \int_0^t \int_{xv} f v_i \frac{\partial \varphi}{\partial x_i} = \\ \int_0^t \int_{xv} (\bar{a}_{ij} + \varepsilon \delta_{ij}) f \frac{\partial^2 \varphi}{\partial v_i \partial v_j} + \int_0^t \int_{xv} \left(\frac{\partial \bar{a}_{ij}}{\partial v_i} + \bar{b}_j(1-2f) \right) f \frac{\partial \varphi}{\partial v_j} \\ + \int_0^t \int_{xv} \left[\left(\frac{\partial \bar{b}_j}{\partial v_j} (1-2f) - 2\bar{b}_j \frac{\partial f}{\partial v_j} \right) f - \frac{\partial \bar{b}_j}{\partial v_j} f(1-f) \right] \varphi$$

for almost every $t > 0$. The integrals in $(0, t)$ being obviously continuous functions of t , imply that for every $\varphi \in \mathcal{D}(\mathbb{R}_{x,v}^{2N})$,

$$\begin{cases} (0, \infty) & \rightarrow \mathbb{R} \\ t & \mapsto \int_{xv} f(t)\varphi(t) \end{cases}$$

is a continuous function. The equality (27) is then valid for every $t > 0$.

Let $\psi \in \mathcal{D}(\mathbb{R}_{x,v}^{2N})$ and choose φ such that $\varphi(t, x, v) = \psi(x, v)$ for every $(t, x, v) \in [0, 1] \times \mathbb{R}_{x,v}^{2N}$ in (27). The above continuity then implies that $f \in C((0, \infty); \mathcal{D}'(\mathbb{R}_{x,v}^{2N}))$. Also, if we take the equation (27) to the limit $t \rightarrow 0^+$, we conclude

$$\int_{xv} f(t)\psi \xrightarrow{t \rightarrow 0^+} \int_{xv} f_0\psi,$$

that is, we have convergence to the initial data and therefore f is a solution in the sense of Definition 1. Finally, the estimates (24) and (25), being uniform in k , are therefore still valid in the limit. \square

We will now show that the conservation laws are approximately obeyed by the solutions of the approximate equation, a result that will be useful for deducing the conservation laws obeyed by the LFD solution we will construct.

Proposition 4. *The approximated solution $f \in C((0, \infty); \mathcal{D}'(\mathbb{R}_{x,v}^{2N})) \cap L^2((0, \infty) \times \mathbb{R}_x^N; H^1(\mathbb{R}_v^N))$ to (16) constructed in Proposition 3 satisfies, for almost every $t > 0$,*

1) *Conservation of mass and linear momentum:*

$$\iint f(t, x, v) dx dv = \iint f_0(x, v) dx dv, \\ \iint f(t, x, v) v_i dx dv = \iint f_0(x, v) v_i dx dv \quad \forall i \in \{1, \dots, N\}.$$

2) *Kinetic energy:*

$$\iint |v|^2 f(t, x, v) dx dv = \iint |v|^2 f_0(x, v) dx dv + 2\varepsilon t \|f_0\|_{L^1}.$$

3) *Moment of inertia:*

$$\iint f(t, x, v) |x - tv|^2 dx dv = \iint f_0(x, v) |x|^2 dx dv + \frac{2\varepsilon t^3}{3} \|f_0\|_{L^1}.$$

4) *Entropy inequality:*

$$\iint s(t, x, v) dx dv + \int_0^t \iint d(t, x, v) \leq \iint s(0, x, v) dx dv,$$

where

$$s(t, x, v) = f \log f + (1 - f) \log(1 - f)$$

and

$$d(t, x, v) = \int a(v - v_*) f(1 - f) f_*(1 - f_*) \left| \frac{\nabla_v f}{f(1 - f)} - \frac{\nabla_{v_*} f_*}{f_*(1 - f_*)} \right|^{\otimes 2} dv_*,$$

with the usual notations $f = f(t, x, v)$ and $f_* = f(t, x, v_*)$.

Proof. As in the proof of Proposition 3 we will notate, for brevity, L^p and L_{loc}^p for $L_{loc}^p((0, \infty); L^p(\mathbb{R}_{x,v}^{2N}))$ and $L_{loc}^p((0, \infty) \times \mathbb{R}_{x,v}^{2N})$, respectively.

Let $(\varphi^n)_n$ be a sequence of smooth truncations, with $\varphi^n(x, v) = \varphi((x, v)/n)$, $\varphi(0) = 1$, $0 \leq \varphi \leq 1$, this way $\text{supp } \varphi^n \subset B_n$ and $\varphi^n \rightarrow 1$ a.e. Let $\psi = \psi(t, x, v)$ be a smooth function such that $e^{-\alpha(|x|^2 + |v|^2)}(|\psi| + |\nabla_{t,x,v} \psi| + |\nabla_{t,x,v}^2 \psi|) \in L^2$ and consider $(f^k)_k$ be the sequence of solutions to (23) constructed in the proof of Proposition 3.

Multiplying equation (23) by $\varphi^n \psi$ we have, after integrating by parts,

$$(28) \quad \int_{xv} f^k(t) \varphi^n \psi(t) - \int_{xv} f_0 \varphi^n \psi_0 - \int_{txv} f^k \left(\frac{\partial(\varphi^n \psi)}{\partial t} + v_i \frac{\partial(\varphi^n \psi)}{\partial x_i} \right) = - \int_{txv} \bar{A}_{ij}^{k-1} \frac{\partial f^k}{\partial v_i} \frac{\partial(\varphi^n \psi)}{\partial v_j} \\ - \int_{txv} \bar{b}_j^{k-1} (1 - 2f^{k-1}) \frac{\partial f^k}{\partial v_j} \varphi^n \psi - \int_{txv} \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1 - f^k) \varphi^n \psi$$

where we note \int_{txv} the integral $\int_0^t \int_{xv}$, depending on t , and $\bar{A}_{ij}^{k-1} = \bar{a}_{ij}^{k-1} + \varepsilon \delta_{ij}$.

Let us pass each of the integrals to the limit $n \rightarrow \infty$. The first two converge to

$$\int_{xv} f^k(t) \psi(t) - \int_{xv} f_0 \psi_0$$

by dominated convergence, given the uniform bounds (19). Next, we have

$$\int_{txv} f^k \left(\frac{\partial(\varphi^n \psi)}{\partial t} + v_i \frac{\partial(\varphi^n \psi)}{\partial x_i} \right) = \frac{1}{n} \int_{(0,t) \times B_n \times B_n} f^k \left(\frac{\partial \varphi}{\partial t} + v_i \frac{\partial \varphi}{\partial x_i} \right) \left(\frac{\cdot}{n} \right) \psi \\ + \int_{txv} f^k \varphi^n \left(\frac{\partial \psi}{\partial t} + v_i \frac{\partial \psi}{\partial x_i} \right).$$

Given the uniform bounds (19), the first integral is bounded independently of n , and thus the first term of the right-hand side vanishes in the limit. The second term converges by dominated convergence and the right-hand side tends to

$$\int_{txv} f^k \left(\frac{\partial \psi}{\partial t} + v_i \frac{\partial \psi}{\partial x_i} \right).$$

Developing the next integral in (28),

$$\int_{txv} \bar{A}_{ij}^{k-1} \frac{\partial f^k}{\partial v_i} \frac{\partial(\varphi^n \psi)}{\partial v_j} = \frac{1}{n} \int_{txv} \bar{A}_{ij}^{k-1} \frac{\partial f^k}{\partial v_i} \frac{\partial \varphi}{\partial v_j} \left(\frac{\cdot}{n} \right) \psi + \int_{txv} \bar{A}_{ij}^{k-1} \frac{\partial f^k}{\partial v_i} \varphi^n \frac{\partial \psi}{\partial v_j}.$$

the first integral is uniformly bounded, and thus the first term converges to zero. Using dominated convergence on the second term, we have that the above expression converges to

$$\int_{txv} \bar{A}_{ij}^{k-1} \frac{\partial f^k}{\partial v_i} \frac{\partial \psi}{\partial v_j}.$$

Once again by dominated convergence, the last two integrals in (28) converge to

$$- \int_{txv} \bar{b}_j^{k-1} (1 - 2f^{k-1}) \frac{\partial f^k}{\partial v_j} \psi - \int_{txv} \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1 - f^k) \psi,$$

thus, in the limit $n \rightarrow \infty$ equation (28) becomes

$$(29) \quad \int_{xv} f^k(t)\psi(t) - \int_{xv} f_0\psi_0 - \int_{txv} f^k \left(\frac{\partial\psi}{\partial t} + v_i \frac{\partial\psi}{\partial x_i} \right) = - \int_{txv} \bar{A}_{ij}^{k-1} \frac{\partial f^k}{\partial v_i} \frac{\partial\psi}{\partial v_j} \\ - \int_{txv} \bar{b}_j^{k-1} (1 - 2f^{k-1}) \frac{\partial f^k}{\partial v_j} \psi - \int_{txv} \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1 - f^k) \psi.$$

We know that $f^k \rightarrow f$ almost everywhere. Since the terms $\bar{A}^k, \bar{b}^k, \frac{\partial \bar{b}^k}{\partial v_i}$ are all uniformly bounded in L^∞ with respect to k , the integrands are all dominated by integrable functions independent of k , by the bound (19), with the exception of

$$\bar{b}_j^{k-1} (1 - 2f^{k-1}) \frac{\partial f^k}{\partial v_j} \psi$$

which contains a derivative of f^k . Nevertheless we have that

$$\left\| \bar{b}_j^{k-1} (1 - 2f^{k-1}) - \bar{b}_j (1 - 2f) \right\|_{L^2} \leq 2 \left\| \bar{b}_j^{k-1} \right\|_{L^\infty} \|f^{k-1} - f\|_{L^2} + \left\| \bar{b}_j^{k-1} - \bar{b}_j \right\|_{L^2}$$

and thus, by the weak-strong result,

$$\bar{b}_j^{k-1} (1 - 2f^{k-1}) \frac{\partial f^k}{\partial v_j} \psi \rightarrow \bar{b}_j (1 - 2f) \frac{\partial f}{\partial v_j} \psi$$

in weak L^1 , which implies the convergence of the integral.

This way, we can pass equation (29) to the limit $k \rightarrow \infty$, which gives

$$\int_{xv} f(t)\psi(t) - \int_{xv} f_0\psi_0 - \int_{txv} f \left(\frac{\partial\psi}{\partial t} + v_i \frac{\partial\psi}{\partial x_i} \right) = - \int_{txv} \bar{A}_{ij} \frac{\partial f}{\partial v_i} \frac{\partial\psi}{\partial v_j} \\ - \int_{txv} \bar{b}_j (1 - 2f) \frac{\partial f}{\partial v_j} \psi - \int_{txv} \frac{\partial \bar{b}_j}{\partial v_j} f (1 - f) \psi.$$

Expanding the convolutions and symmetrizing the collision integral, we have

$$\int_{xv} f(t)\psi - \int_{xv} f_0\psi_0 = \int_0^t \int_{xv} f \left(\frac{\partial\psi}{\partial t} + v_i \frac{\partial\psi}{\partial x_i} \right) + \varepsilon \int_0^t \int_{xv} f \Delta_v \psi \\ - \int_0^t \int_{xvv_*} a_{ij} (v - v_*) \left(f_*(1 - f_*) \frac{\partial f}{\partial v_j} - f(1 - f) \frac{\partial f_*}{\partial v_{*,j}} \right) \left(\frac{\partial\psi}{\partial v_i} - \frac{\partial\psi_*}{\partial v_{*,i}} \right).$$

Taking $\psi = 1$ we have the conservation of mass and for $\psi = v_k$ we have the conservation of linear momentum. For $\psi = |v|^2$, it follows, since a satisfies (15),

$$\int_{xv} f(t)|v|^2 - \int_{xv} f_0|v|^2 = 2\varepsilon \int_0^t \int_{xv} f = 2\varepsilon t \int_{xv} f_0.$$

Next, for $\psi = |x - tv|^2$, we also have from (15) that

$$\int_{xv} f(t)|x - tv|^2 - \int_{xv} f_0|x|^2 = 2\varepsilon \int_0^t t^2 \int_{xv} f = \frac{2\varepsilon t^3}{3} \int_{xv} f_0.$$

Now we pass to the proof of the entropy inequality. Multiplying (23) by $\log\left(\frac{f^k}{1-f^k}\right)$ and integrating in x, v leads,

$$\begin{aligned} \int_{xv} \frac{\partial f^k}{\partial t} \log\left(\frac{f^k}{1-f^k}\right) + \int_{xv} v_i \frac{\partial f^k}{\partial x_i} \log\left(\frac{f^k}{1-f^k}\right) &= \int_{xv} \frac{\partial}{\partial v_j} \left\{ \bar{a}_{ij}^{k-1} \frac{\partial f^k}{\partial v_i} \right\} \log\left(\frac{f^k}{1-f^k}\right) \\ &\quad - \int_{xv} \bar{b}_j^{k-1} (1-2f^{k-1}) \frac{\partial f^k}{\partial v_j} \log\left(\frac{f^k}{1-f^k}\right) \\ &\quad - \int_{xv} \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1-f^k) \log\left(\frac{f^k}{1-f^k}\right) + \varepsilon \int_{xv} \Delta_v f^k \log\left(\frac{f^k}{1-f^k}\right). \end{aligned}$$

The first integral in the left-hand side equals

$$\frac{d}{dt} \int_{xv} (f^k \log f^k + (1-f^k) \log(1-f^k)).$$

The second one, from the decay estimates for f^k leads

$$\int_{xv} \frac{\partial}{\partial x_i} [v_i (f^k \log f^k + (1-f^k) \log(1-f^k))] = 0.$$

Integrating by parts, the right-hand side becomes

$$\begin{aligned} - \int_{xv} \bar{a}_{ij}^{k-1} \frac{1}{f^k(1-f^k)} \frac{\partial f^k}{\partial v_i} \frac{\partial f^k}{\partial v_j} - \int_{xv} \bar{b}_j^{k-1} (1-2f^{k-1}) \frac{\partial f^k}{\partial v_j} \log\left(\frac{f^k}{1-f^k}\right) \\ - \int_{xv} \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1-f^k) \log\left(\frac{f^k}{1-f^k}\right) - \varepsilon \int_{xv} \frac{|\nabla_v f^k|^2}{f^k(1-f^k)} \end{aligned}$$

which is smaller than

$$\begin{aligned} (30) \quad & - \int_{xv} \bar{a}_{ij}^{k-1} \frac{1}{f^k(1-f^k)} \frac{\partial f^k}{\partial v_i} \frac{\partial f^k}{\partial v_j} - \int_{xv} \bar{b}_j^{k-1} (1-2f^{k-1}) \frac{\partial f^k}{\partial v_j} \log\left(\frac{f^k}{1-f^k}\right) \\ & - \int_{xv} \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1-f^k) \log\left(\frac{f^k}{1-f^k}\right). \end{aligned}$$

Finally, integrating in $t \in (0, T)$ we end up with

$$\begin{aligned} (31) \quad & \int_{xv} s(f^k(t)) - \int_{xv} s(f_0) \leq - \int_{txv} \bar{a}_{ij}^{k-1} \frac{1}{f^k(1-f^k)} \frac{\partial f^k}{\partial v_i} \frac{\partial f^k}{\partial v_j} \\ & - \int_{txv} \bar{b}_j^{k-1} (1-2f^{k-1}) \frac{\partial f^k}{\partial v_j} \log\left(\frac{f^k}{1-f^k}\right) - \int_{txv} \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1-f^k) \log\left(\frac{f^k}{1-f^k}\right), \end{aligned}$$

where $s(x) = x \log x + (1-x) \log(1-x)$.

Let's pass each of these integrals to the limit. Starting with the right-hand side, notice that since \bar{a}^k converges in L^1 there exists a function $F \in L^1$ such that $|\bar{a}_{ij}^k| \leq F$ for every k , up to extraction of a subsequence. By the equivalence of matrix norms, we conclude that there exists a constant $C > 0$ such that $(\sqrt{\bar{a}^k})_{ij} \leq CF^{1/2}$, where $\sqrt{\bar{a}^k}$ is the (matrix) square root of \bar{a}^k .

Indeed, let $|\cdot|_p$ be the l^p norm in finite dimension. We have, for every $v \in \mathbb{R}^N$ and positive semi-definite matrix A ,

$$|\sqrt{A}v|_2 = \langle \sqrt{A}v, \sqrt{A}v \rangle_2 = \langle Av, v \rangle_2 \leq |Av|_2 |v|_2.$$

Denote $\|M\|_{op} = \sup_{v \in \mathbb{R}^N} \frac{|Mv|_2}{|v|_2}$ the operator norm of a matrix M . The above inequality then implies $\|\sqrt{a^k}\|_{op}^2 \leq \|\bar{a}^k\|_{op}$. Since all norms in finite dimension are equivalent, there exist a $C > 0$ such that $|\sqrt{a^k}|_\infty^2 \leq C|\bar{a}^k|_\infty$, that is, $(\sqrt{a^k})_{ij} \leq C^{1/2}F^{1/2}$, for every $k \in \mathbb{N}$.

The bounds on f^k imply that

$$f^k(1 - f^k) \geq \frac{C_1 e^{-\alpha(|x|^2 + |v|^2)}}{1 + C_2 e^{-\alpha(|x|^2 + |v|^2)}} \geq \frac{C_1}{1 + C_2} e^{-\alpha(|x|^2 + |v|^2)}$$

and thus

$$\left| (\sqrt{a^{k-1}})_{ij} \frac{1}{\sqrt{f^k(1 - f^k)}} e^{-\alpha/2(|x|^2 + |v|^2)} \right| \leq C^{1/2} \sqrt{\frac{1 + C_2}{C_1}} F^{1/2},$$

which is an L^2 function. Also, we have that \bar{a}_{ij}^k converges to \bar{a}_{ij} and f^k to f almost everywhere, which implies, by dominated convergence, that

$$(\sqrt{a^{k-1}})_{ij} \frac{1}{\sqrt{f^k(1 - f^k)}} e^{-\alpha/2(|x|^2 + |v|^2)} \rightarrow (\sqrt{a})_{ij} \frac{1}{\sqrt{f(1 - f)}} e^{-\alpha/2(|x|^2 + |v|^2)} \text{ in } L^2.$$

The bound of the derivative give us that, passing to a subsequence if necessary,

$$(32) \quad e^{\alpha/2(|x|^2 + |v|^2)} \frac{\partial f^k}{\partial v_i} \rightharpoonup e^{\alpha/2(|x|^2 + |v|^2)} \frac{\partial f}{\partial v_i} \text{ in } L^2$$

and thus by weak-strong convergence,

$$(\sqrt{a^{k-1}})_{ij} \frac{1}{\sqrt{f^k(1 - f^k)}} \frac{\partial f^k}{\partial v_i} \rightarrow (\sqrt{a})_{ij} \frac{1}{\sqrt{f(1 - f)}} \frac{\partial f}{\partial v_i} \text{ in } L^2.$$

The lower semi-continuity of the norm under weak convergence implies

$$\left\| (\sqrt{a})_{ij} \frac{1}{\sqrt{f(1 - f)}} \frac{\partial f}{\partial v_i} \right\|_{L^2}^2 \leq \liminf_{k \rightarrow \infty} \left\| (\sqrt{a^{k-1}})_{ij} \frac{1}{\sqrt{f^k(1 - f^k)}} \frac{\partial f^k}{\partial v_i} \right\|_{L^2}^2$$

but since

$$\left| (\sqrt{a})_{ij} \frac{1}{\sqrt{f(1 - f)}} \frac{\partial f}{\partial v_i} \right|^2 = \bar{a}_{ij} \frac{1}{f(1 - f)} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j},$$

we have that

$$(33) \quad \int_{txv} \bar{a}_{ij} \frac{1}{f(1 - f)} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} \leq \liminf_{k \rightarrow \infty} \int_{txv} \bar{a}_{ij}^{k-1} \frac{1}{f^k(1 - f^k)} \frac{\partial f^k}{\partial v_i} \frac{\partial f^k}{\partial v_j}.$$

For the second integral, the bounds on f^k give us that

$$\frac{f^k}{1 - f^k} \leq C_2 e^{-\alpha(|x|^2 + |v|^2)}$$

and thus

$$(34) \quad \left| \log \left(\frac{f^k}{1 - f^k} \right) \right| \leq C e^{\alpha/2(|x|^2 + |v|^2)}$$

for some $C > 0$. We have that \bar{b}_j^{k-1} converges almost everywhere and in L^2 to \bar{b}_j , and is thus dominated by an L^2 function F_2 . But since

$$\left| \bar{b}_j^{k-1} (1 - 2f^{k-1}) e^{-\alpha/2(|x|^2 + |v|^2)} \log \left(\frac{f^k}{1 - f^k} \right) \right| \leq C F_2,$$

it follows that the term in the absolute value converges in L^2 . This together with the convergence (32) leads

$$\int_{txv} \bar{b}_j^{k-1} (1 - 2f^{k-1}) \frac{\partial f^k}{\partial v_j} \log \left(\frac{f^k}{1 - f^k} \right) \rightarrow \int_{txv} \bar{b}_j (1 - 2f) \frac{\partial f}{\partial v_j} \log \left(\frac{f}{1 - f} \right).$$

Finally, for the last integral,

$$\int_{txv} \frac{\partial \bar{b}_j^{k-1}}{\partial v_j} f^{k-1} (1 - f^k) \log \left(\frac{f^k}{1 - f^k} \right),$$

we use that $\frac{\partial \bar{b}_j^{k-1}}{\partial v_j}$ is uniformly bounded in L^∞ . Using the bounds we have on f^k , uniform in k , we have

$$\left| f^{k-1} \log \left(\frac{f^k}{1 - f^k} \right) \right| \leq C e^{-\alpha/2(|x|^2 + |v|^2)}$$

and thus the integrand is uniformly bounded by $C' e^{-\alpha/2(|x|^2 + |v|^2)}$, which is integrable. Since this integrand converges almost everywhere, we have by dominated convergence that the integral converges to

$$\int_{txv} \frac{\partial \bar{b}_j}{\partial v_j} f (1 - f) \log \left(\frac{f}{1 - f} \right).$$

Finally, let us pass the integrals in the left-hand side of (31) to the limit. We have that $s(f^k(t, x, v)) \rightarrow s(f(t, x, v))$ for a.e. $(t, x, v) \in (0, T) \times \mathbb{R}_{x,v}^{2N}$. Also, since

$$|s(f^k)| \leq \left| f^k \log \left(\frac{f^k}{1 - f^k} \right) \right| + |\log(1 - f^k)| \leq |f^k| \left(1 + \left| \log \left(\frac{f^k}{1 - f^k} \right) \right| \right)$$

we have, from (34) and the uniform bound on f^k ,

$$|s(f^k)| \leq C' e^{-\alpha/2(|x|^2 + |v|^2)},$$

for some $C' > 0$, and thus the left-hand side converges, for a.e. $t \in (0, T)$, to

$$\int_{xv} s(f(t)) - \int_{xv} s(f_0).$$

Finally, taking the \liminf in (31), it follows

$$\begin{aligned} \int_{xv} s(f(t)) - \int_{xv} s(f_0) &\leq - \liminf_{k \rightarrow \infty} \int_{txv} \bar{a}_{ij}^{k-1} \frac{1}{f^k (1 - f^k)} \frac{\partial f^k}{\partial v_i} \frac{\partial f^k}{\partial v_j} \\ &\quad - \int_{txv} \bar{b}_j (1 - 2f) \frac{\partial f}{\partial v_j} \log \left(\frac{f}{1 - f} \right) - \int_{txv} \frac{\partial \bar{b}_j}{\partial v_j} f (1 - f) \log \left(\frac{f}{1 - f} \right). \end{aligned}$$

which, from (33), is smaller than

$$(35) \quad - \int_{txv} \bar{a}_{ij} \frac{1}{f(1 - f)} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} - \int_{txv} \bar{b}_j (1 - 2f) \frac{\partial f}{\partial v_j} \log \left(\frac{f}{1 - f} \right) - \int_{txv} \frac{\partial \bar{b}_j}{\partial v_j} f (1 - f) \log \left(\frac{f}{1 - f} \right).$$

Since $\{x \mapsto x(1 - x)\}$ is a smooth function and f is bounded, we have that

$$(1 - 2f) \frac{\partial f}{\partial v_j} = \frac{\partial(f(1 - f))}{\partial v_j}$$

and since b_j is smooth in the v variable, we can justify the product rule with $f(1 - f)$, leading

$$\bar{b}_j \frac{\partial(f(1 - f))}{\partial v_j} + \frac{\partial \bar{b}_j}{\partial v_j} f(1 - f) = \frac{\partial(\bar{b}_j f(1 - f))}{\partial v_j},$$

then (35) equals

$$-\int_{txv} \bar{a}_{ij} \frac{1}{f(1-f)} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} - \int_{txv} \frac{\partial(\bar{b}_j f(1-f))}{\partial v_j} \log\left(\frac{f}{1-f}\right).$$

Now, notice that $\log\left(\frac{f}{1-f}\right)$ is H^1 in the v variable, since the uniform bounds (19) imply, after passing to the limit in k ,

$$\frac{1}{f(1-f)} \leq C'' e^{\alpha(|x|^2+|v|^2)},$$

for some $C'' > 0$. Then the bound (20), which is still valid in the limit $k \rightarrow \infty$ by (32), implies we can justify the chain rule,

$$\frac{\partial}{\partial v_i} \left[\log\left(\frac{f}{1-f}\right) \right] = \frac{1}{f(1-f)} \frac{\partial f}{\partial v_i}$$

is bounded in L^2 . Thus finally, by the definition of weak derivative, (35) equals

$$-\int_{txv} \bar{a}_{ij} \frac{1}{f(1-f)} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} + \int_{txv} \bar{b}_j \frac{\partial f}{\partial v_j}.$$

Then, expanding the convolutions, this equals

$$-\int_{txv} a_{ij}(v-v_*) \left(\frac{f_*(1-f_*)}{f(1-f)} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} - \frac{\partial f_*}{\partial v_{*,i}} \frac{\partial f}{\partial v_j} \right)$$

which, by symmetrization and (15), leads

$$-\frac{1}{2} \int_{txv} a(v-v_*) f(1-f) f_*(1-f_*) \left| \frac{\nabla_v f}{f(1-f)} - \frac{\nabla_{v_*} f_*}{f_*(1-f_*)} \right|^{\otimes 2},$$

the desired dissipation. \square

5. THE EXISTENCE THEOREM

Now we will use the Proposition 1 to prove Theorem 1. We've divided this proof into a few steps. First, we will construct a sequence of initial data and approximate collision kernels, thus obtaining approximate solutions using Proposition 3. Then, in a second step we will use these solutions together with Proposition 1 to construct a global solution for LFD. Finally, we will prove that this solution obeys the conservation laws stated in Theorem 1, as well as the entropy inequality.

Proof of Theorem 1. Part 1: Approximated initial data and collision kernels

We start by defining a sequence of approximated initial data. Take f_0 such that $f_0(1+|x|^2+|v|^2) \in L^1(\mathbb{R}_{x,v}^{2N})$ and $0 \leq f_0 \leq 1$. Let $\chi \in C_c^\infty(\mathbb{R}_{x,v}^{2N})$ be a positive function such that $\int_{xv} \chi = 1$ and consider $\chi^n(x, v) = n^{2N} \chi(nx, nv)$. Then, the convolution satisfies $0 \leq f_0 * \chi^n \leq 1$.

Also let $\varphi \in C_c^\infty(\mathbb{R}_{x,v}^{2N})$ be a radially decreasing function such that $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, $\text{supp } \varphi \subset B_1$ and consider $\varphi^n(x, v) = \varphi\left(\frac{x}{n}, \frac{v}{n}\right)$. This way, φ^n is an increasing sequence of positive functions in $C_c^\infty(\mathbb{R}_{x,v}^{2N})$ such that $\varphi^n \rightarrow 1$ pointwise. Therefore, $(f_0 * \chi^n) \varphi^n$ is a sequence of $C_c^\infty(\mathbb{R}_{x,v}^{2N})$ which tends to f_0 almost everywhere, which in turn implies that the sequence of functions

$$f_0^n = \frac{\frac{1}{n} e^{-(|x|^2+|v|^2)} + (f_0 * \chi^n) \varphi^n}{1 + \frac{2}{n}}$$

converges a.e. to f_0 and satisfies $0 < f_0^n < 1$.

Let us show that

$$(36) \quad f_0^n(1 + |x|^2 + |v|^2) \rightarrow f_0(1 + |x|^2 + |v|^2) \text{ in } L^1(\mathbb{R}_{x,v}^{2N}),$$

Indeed, writing $\langle x, v \rangle = (1 + |x|^2 + |v|^2)$, we have

$$\begin{aligned} \int_{xv} |f_0^n - f_0| \langle x, v \rangle &\leq \frac{1}{n+2} \int_{xv} e^{-(|x|^2+|v|^2)} \langle x, v \rangle + \frac{2}{n+2} \int_{xv} f_0 \langle x, v \rangle \\ &\quad + \frac{n}{n+2} \int_{xv} (f_0 * \chi^n) |1 - \varphi^n| \langle x, v \rangle + \frac{n}{n+2} \int_{xv} |(f_0 * \chi^n) - f_0| \langle x, v \rangle. \end{aligned}$$

The first two terms in the right-hand side converge to 0 and the third is bounded by

$$\int_{xv} (1 - \varphi^n) \langle x, v \rangle,$$

which converges to 0 by Beppo-Levi. Hence, we just need to show that

$$(f_0 * \chi^n) \langle x, v \rangle \rightarrow f_0 \langle x, v \rangle \text{ in } L^1(\mathbb{R}_{x,v}^{2N}).$$

We write

$$\int_{xv} |(f_0 * \chi^n) - f_0| \langle x, v \rangle \leq \int_{xv} |(f_0 * \chi^n) \langle x, v \rangle - (f_0 \langle x, v \rangle) * \chi^n| + \int_{xv} |(f_0 \langle x, v \rangle) * \chi^n - f_0 \langle x, v \rangle|$$

The second integral converges to zero by standard approximation by convolution theorems, since $f_0 \langle x, v \rangle \in L^1$. For the first one, using that $|x|^2 \leq |x - x'|^2 + |x'|^2$,

$$\begin{aligned} |(f_0 * \chi^n) \langle x, v \rangle - (f_0 \langle x, v \rangle) * \chi^n| \langle x, v \rangle &\leq \left| \int \chi^n(x - x', v - v') f_0(x', v') \langle x, v \rangle - \chi^n(x - x', v - v') f_0(x', v') \langle x', v' \rangle dx' dv' \right| \\ &\leq \left| \int \chi^n(x - x', v - v') f_0(x', v') (|x|^2 + |v|^2 - |x'|^2 - |v'|^2) dx' dv' \right| \\ &\leq \int \chi^n(x - x', v - v') (|x - x'|^2 + |v - v'|^2) f_0(x', v') dx' dv' \\ &= [(\chi^n \langle x, v \rangle) * f_0](x, v). \end{aligned}$$

Hence by Young's inequality it follows that the first integral is bounded by $\|\chi^n \langle x, v \rangle\|_{L^1} \|f_0\|_{L^1}$, but since $\|\chi^n \langle x, v \rangle\|_{L^1} = \frac{1}{n^2} \|\chi \langle x, v \rangle\|_{L^1}$, it must converge to zero and therefore we have $f_0^n \langle x, v \rangle \rightarrow f_0 \langle x, v \rangle$ in $L^1(\mathbb{R}_{x,v}^{2N})$.

Let us check that these f_0^n have the decay properties for us to apply Proposition 3. We have, trivially,

$$f_0^n \geq \frac{1}{n+2} e^{-(|x|^2+|v|^2)}.$$

On the other hand note that for $C, \alpha > 0$,

$$g \leq \frac{C e^{-\alpha(|x|^2+|v|^2)}}{1 + C e^{-\alpha(|x|^2+|v|^2)}} \iff \frac{g}{1-g} \leq C e^{-\alpha(|x|^2+|v|^2)}.$$

So, let $C_n > 0$ be such that $(f_0 * \chi^n) \varphi^n \leq C_n e^{-(|x|^2+|v|^2)}$. We have that

$$\frac{f_0^n}{1 - f_0^n} = \frac{\frac{1}{n} e^{-(|x|^2+|v|^2)} + (f_0 * \chi^n) \varphi^n}{1 + \frac{2}{n} - \frac{1}{n} e^{-(|x|^2+|v|^2)} - (f_0 * \chi^n) \varphi^n}$$

but since $1 + \frac{2}{n} - \frac{1}{n}e^{-(|x|^2+|v|^2)} - (f_0 * \chi^n)\varphi^n \geq \frac{1}{n}$, it follows

$$\frac{f_0^n}{1 - f_0^n} \leq e^{-(|x|^2+|v|^2)} + n(f_0 * \chi^n)\varphi^n \leq (nC_n + 1)e^{-(|x|^2+|v|^2)},$$

which shows f_0^n has the desired decay.

We now approximate the collision kernel of the equation. In order to apply Proposition 1 to solutions constructed with Proposition 3, we need to find a sequence $a^n \in \mathcal{S}(\mathbb{R}^N)$ satisfying property (15) for each n and such that $\bar{A}_{ij}^n \equiv a^n *_{\nu} (f^n(1 - f^n))$ satisfies (9).

Let $P(z) = \left(I - \frac{z \otimes z}{|z|^2}\right)$ and consider its approximation

$$(37) \quad P^n(z) = \frac{|z|^2}{|z|^2 + 1/n} I - \frac{z \otimes z}{|z|^2 + 1/n}.$$

Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be a positive, radial function such that $\text{supp } \eta \subset [-1, 1]$, $\int \eta dz = 1$ and consider $\eta_n(z) = n^N \eta(nz)$, a radially symmetric mollifier in $C_c^\infty(\mathbb{R}^N)$ with $\text{supp } \eta_n \subset B_{1/n}$. Also, let $\psi \in \mathcal{S}(\mathbb{R}^N)$ be a radial function such that $\psi > 0$ and $\psi(0) = 1$, in such a way that $\psi^n(z) = \psi(z/n)$ converges almost everywhere to 1.

We denote the mollification $\Gamma^n(z) = [\Gamma(|\cdot|) * \eta_n](z)$, which is radial as convolution of radially symmetric functions, and converges to $\Gamma(|z|)$ in $L^r(\mathbb{R}^N) + L^{\infty,*}(\mathbb{R}^N)$. Passing to a subsequence in n , we can suppose that Γ^n is dominated by an $L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ function. The collision kernel is then approximated as

$$(38) \quad a^n(z) = \Gamma^n(z) \psi^n(z) P^n(z).$$

Notice this is indeed a symmetric matrix satisfying (15). Also, for each n we have that $a^n \in \mathcal{S}(\mathbb{R}^N)$, as the product of $C^\infty(\mathbb{R}^N)$, bounded functions with a function in $\mathcal{S}(\mathbb{R}^N)$.

Next, we show that this approximation preserves the quasi-ellipticity of the matrix $a(z)$. Notice that our approximation P^n is such that $P^n(z) \geq P(z)$ for every $n \in \mathbb{N}$ and every $z \in \mathbb{R}^N$, hence

$$a^n(z) \geq \Gamma^n(z) \psi^n(z) P(z).$$

Now, let $0 < R < 1$ and notice that $\text{supp } \eta_n \subset B_1$ for every n . If $|z| \leq R$ then from (5) we conclude

$$[\Gamma(|\cdot|) * \eta_n](z) = \int \Gamma(|z - z_*|) \eta_n(z_*) dz_* \geq K_{2R},$$

and since ψ_n is also lower bounded in this set, we have that for every $R > 0$ there exists some constant $C_R > 0$ such that

$$(39) \quad a^n(z) \geq C_R \left(I - \frac{z \otimes z}{|z|^2}\right), \quad \forall |z| \leq R.$$

The last property we want to show about this approximation is that a_{ij}^n and $\frac{\partial a_{ij}^n}{\partial z_j}$ converge in $L^1(\mathbb{R}^N) + L^{\infty,*}(\mathbb{R}^N)$. We say that a sequence Ψ^n converges in $L^p(\mathbb{R}^N) + L^{\infty,*}(\mathbb{R}^N)$, for some $1 \leq p \leq \infty$, if we can decompose it as $\alpha^n + \beta^n$, where α^n converges strongly $L^p(\mathbb{R}^N)$ and β^n converges *-weakly in $L^\infty(\mathbb{R}^N)$, that is,

$$\begin{array}{ccccc} \Psi^n(z) & = & \alpha^n(z) & + & \beta^n(z) \\ & & \downarrow L^p(\mathbb{R}^N) & & \downarrow L^\infty(\mathbb{R}^N) \\ & & \alpha(z) & + & \beta(z) & = & \Psi(z). \end{array}$$

The convergence of a^n follows from standard theorems of regularization by convolution. Indeed, the right-hand side of (38) converges almost everywhere to $\Gamma(|z|)P(z)$ and, up to extraction of a subsequence in n , we have that it's dominated by a function in $L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \subset L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, therefore converging in $L^1(\mathbb{R}^N) + L^{\infty,*}(\mathbb{R}^N)$ by dominated convergence.

For $\frac{\partial a_{ij}^n}{\partial z_j}$ we have, differentiating (38),

$$(40) \quad \frac{\partial a_{ij}^n}{\partial z_j}(z) = \frac{\partial}{\partial z_j} \left[\Gamma^n(z) P_{ij}^n(z) \right] \psi^n(z) + \Gamma^n(z) P_{ij}^n(z) \frac{\partial \psi^n}{\partial z_j}(z).$$

Now notice that if $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ is a smooth, radial function then we have that for every $\varepsilon > 0$,

$$(41) \quad \frac{\partial}{\partial z_j} \left[\Psi(z) \left(\frac{|z|^2}{|z|^2 + \varepsilon} \delta_{ij} - \frac{z_i z_j}{|z|^2 + \varepsilon} \right) \right] = \Psi(z) \xi_\varepsilon(|z|) \frac{z_i}{|z|^2},$$

where

$$\xi_\varepsilon(|z|) = -(N-3) \frac{|z|^2}{|z|^2 + \varepsilon}.$$

Hence it follows that

$$\frac{\partial}{\partial z_j} \left[\Gamma^n(z) P_{ij}^n(z) \right] \psi^n(z) = \Gamma^n(z) \xi_n(|z|) \psi^n(z) \frac{z_i}{|z|^2},$$

which is dominated in $L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, therefore converges in $L^1(\mathbb{R}^N) + L^{\infty,*}(\mathbb{R}^N)$ to

$$-\Gamma(|z|)(N-3) \frac{z_i}{|z|^2} = \frac{\partial a_{ij}}{\partial z_j}.$$

Meanwhile, the second term of (40) equals

$$\Gamma^n(z) P_{ij}^n(z) \frac{1}{n} \frac{\partial \psi}{\partial z_j} \left(\frac{z}{n} \right) \frac{z_i}{|z|},$$

and similarly one can see that this converges to 0 in $L^1(\mathbb{R}^N) + L^{\infty,*}(\mathbb{R}^N)$. Hence, we have $\frac{\partial a_{ij}^n}{\partial z_j} \rightarrow \frac{\partial a_{ij}}{\partial z_j}$ in $L^1(\mathbb{R}^N) + L^{\infty,*}(\mathbb{R}^N)$.

In this next step, we use the approximate data to obtain approximate solutions using Proposition 3, for which we can obtain compactness using Proposition 1.

Part 2: Approximate solutions

Let $\varepsilon_n \rightarrow 0$. For each n , let f^n to be the global solution to (16) given by Proposition 3 using the a^n and the f_0^n constructed in Part 1. We have, for every $\varphi \in \mathcal{D}((0, \infty) \times \mathbb{R}_{x,v}^{2N})$,

$$(42) \quad \begin{aligned} \int_{xv} f^n(t) \varphi(t) - \int_0^t \int_{xv} f^n \frac{\partial \varphi}{\partial t} - \int_0^t \int_{xv} f^n v_i \frac{\partial \varphi}{\partial x_i} &= \int_0^t \int_{xv} (\bar{a}_{ij}^n + \varepsilon_n \delta_{ij}) f^n \frac{\partial^2 \varphi}{\partial v_i \partial v_j} \\ &+ \int_0^t \int_{xv} \left(\frac{\partial \bar{a}_{ij}^n}{\partial v_i} + \bar{b}_j^n (1 - 2f^n) \right) f^n \frac{\partial \varphi}{\partial v_j} \\ &+ \int_0^t \int_{xv} \left[\left(\frac{\partial \bar{b}_j^n}{\partial v_j} (1 - 2f^n) - 2\bar{b}_j^n \frac{\partial f^n}{\partial v_j} \right) f^n - \frac{\partial \bar{b}_j^n}{\partial v_j} f^n (1 - f^n) \right] \varphi, \end{aligned}$$

for every $t > 0$. This cannot be applied directly to Proposition 1, because the form of the equation is not yet compatible. Given the $H_{v,loc}^1$ regularity of f^n and \bar{b}^n , we can justify

$$\int_{txv} \left(\frac{\partial \bar{b}_j^n}{\partial v_j} (1 - 2f^n) - 2\bar{b}_j^n \frac{\partial f^n}{\partial v_j} \right) f^n \varphi = \int_{txv} \frac{\partial}{\partial v_j} \left(\bar{b}_j^n (1 - 2f^n) \right) f^n \varphi,$$

where we have notated \int_{txv} the integrals $\int_0^t \int_{xv} dt$. By definition of weak derivative, we have

$$\begin{aligned} \int_{txv} \frac{\partial}{\partial v_j} \left(\bar{b}_j^n (1 - 2f^n) \right) f^n \varphi &= - \int_{txv} \bar{b}_j^n (1 - 2f^n) \frac{\partial(f^n \varphi)}{\partial v_j} \\ &= - \int_{txv} \bar{b}_j^n (1 - 2f^n) \frac{\partial f^n}{\partial v_j} \varphi - \int_{txv} \bar{b}_j^n (1 - 2f^n) f^n \frac{\partial \varphi}{\partial v_j}. \end{aligned}$$

Now, once again the local regularity of f^n in the variable v allows us to write

$$\frac{\partial f^n (1 - f^n)}{\partial v_j} = (1 - 2f^n) \frac{\partial f^n}{\partial v_j}$$

and thus the first integral above equals, by definition of weak derivative,

$$- \int_{txv} \bar{b}_j^n \frac{\partial(f^n (1 - f^n))}{\partial v_j} \varphi = \int_{txv} f^n (1 - f^n) \frac{\partial \bar{b}_j^n}{\partial v_j} \varphi + \int_{txv} f^n (1 - f^n) \bar{b}_j^n \frac{\partial \varphi}{\partial v_j},$$

where we have used the v regularity of φ and \bar{b} to justify the product rule. We thus have, finally

$$\int_{txv} \left[\left(\frac{\partial \bar{b}_j^n}{\partial v_j} (1 - 2f^n) - 2\bar{b}_j^n \frac{\partial f^n}{\partial v_j} \right) f^n - \frac{\partial \bar{b}_j^n}{\partial v_j} f^n (1 - f^n) \right] \varphi = \int_{txv} \bar{b}_j^n (f^n)^2 \frac{\partial \varphi}{\partial v_j}.$$

Substituting this in (42) leads

$$\begin{aligned} (43) \quad \int_{xv} f^n(t) \varphi(t) - \int_{txv} f^n \frac{\partial \varphi}{\partial t} - \int_{txv} f^n v_i \frac{\partial \varphi}{\partial x_i} &= \int_{txv} (\bar{a}_{ij}^n + \varepsilon_n \delta_{ij}) f^n \frac{\partial^2 \varphi}{\partial v_i \partial v_j} \\ &\quad + \int_{txv} \left(\frac{\partial \bar{a}_{ij}^n}{\partial v_i} f^n + \bar{b}_j^n f^n (1 - f^n) \right) \frac{\partial \varphi}{\partial v_j}, \end{aligned}$$

which is then in the form (8). Given the approximate a priori estimates from Proposition 4, this sequence satisfies all the hypothesis for Proposition 1 and thus we have $L_{loc}^1((0, \infty); L^1(\mathbb{R}_{x,v}^{2N}))$ compactness for the sequence f^n . Passing to a subsequence we then have

$$(44) \quad f^n \rightarrow f \text{ in } L_{loc}^1((0, \infty); L^1(\mathbb{R}_{x,v}^{2N})) \text{ and a.e. in } (0, \infty) \times \mathbb{R}_{x,v}^{2N}.$$

Part 3: Passing the equation to the limit

We start by studying the convergence of the coefficients of the equation. For this, let us recall that if $A^n \rightarrow A$ in $L^1(\mathbb{R}^N) + L^{\infty,*}(\mathbb{R}^N)$ and $F^n \rightarrow F$ in $L_{loc}^1((0, \infty); L^1(\mathbb{R}_{x,v}^{2N}))$, then one can show that

$$A^n *_v F^n \rightarrow A *_v F \text{ in } L_{loc}^1((0, \infty) \times \mathbb{R}_{x,v}^{2N}).$$

Therefore, we have

$$\bar{a}_{ij}^n = a_{ij}^n *_v (f^n (1 - f^n)) \rightarrow a_{ij} *_v (f (1 - f)) \equiv \bar{a}_{ij}$$

in $L_{loc}^1((0, \infty) \times \mathbb{R}_{x,v}^{2N})$, for every $i, j \in \{1, 2, \dots, N\}$ and also that

$$\begin{aligned} \frac{\partial \bar{a}_{ij}^n}{\partial v_j} &\equiv \bar{a}_{ij} = \left(\frac{\partial a_{ij}^n}{\partial v_j} \right) *_v (f^n (1 - f^n)) \rightarrow \left(\frac{\partial a_{ij}}{\partial v_j} \right) *_v (f (1 - f)) \equiv \frac{\partial \bar{a}_{ij}}{\partial v_j} \\ \bar{b}_i^n &= \frac{\partial a_{ij}^n}{\partial v_j} *_v f^n \rightarrow \frac{\partial a_{ij}}{\partial v_j} *_v f \equiv \bar{b}_i \end{aligned}$$

in $L_{loc}^1((0, \infty) \times \mathbb{R}_{x,v}^{2N})$, for every $i \in \{1, 2, \dots, N\}$.

We now show that this convergence is sufficient to pass equation (43) to the limit in the sense of distributions. By dominated convergence, the first three integrals of (43) converge, respectively, to

$$\int_{xv} f(t)\varphi(t), \int_{txv} f \frac{\partial \varphi}{\partial t} \text{ and } \int_{txv} f v_i \frac{\partial \varphi}{\partial x_i},$$

for almost every $t > 0$.

For the two integrals in the right-hand side of (43), we begin by noticing that $f^n \xrightarrow{*} f$ and $f^n(1 - f^n) \xrightarrow{*} f(1 - f)$ in $L^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$, from dominated convergence. Next, recall that if $A^n \rightarrow A$ in $L^1_{loc}((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ and $F^n \xrightarrow{*} F$ in $L^\infty((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ then we have $A^n F^n \rightarrow AF$ in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ and therefore

$$\bar{a}_{ij}^n f^n \rightarrow \bar{a}_{ij} f \text{ and } \frac{\partial \bar{a}_{ij}^n}{\partial v_j} f^n + \bar{b}_i^n f^n(1 - f^n) \rightarrow \frac{\partial \bar{a}_{ij}}{\partial v_j} f + \bar{b}_i f(1 - f)$$

in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{x,v}^{2N})$ and we can pass the last two integrals of (43) to the limit, implying that f is a global weak solution of the LFD equation.

Part 4: A priori estimates

We will now show that the weak solution constructed above obeys the a priori estimates. The Pauli exclusion principle is clearly valid for the limit function f , given the a.e. convergence in (44), and therefore f satisfies part 1) of Theorem 1.

Next, the $L^1_{loc}((0, \infty); L^1(\mathbb{R}_{x,v}^{2N}))$ convergence implies (possibly passing to a subsequence) that we have $f^n(t) \rightarrow f(t)$ in $L^1(\mathbb{R}_{x,v}^{2N})$, for almost every $t \in (0, \infty)$. The conservation of mass for f^n , given by Proposition 4, tells us that

$$\int_{xv} f^n(t) = \int_{xv} f_0^n \text{ for a.e. } t \in (0, \infty).$$

The integral in the left-hand side then converges to $\int_{xv} f(t)$ and from (36) the right-hand side converges to $\int_{xv} f_0$ thus obtaining the conservation of mass in the limit. In particular, this implies that $f \in L^\infty((0, \infty); L^1(\mathbb{R}_{x,v}^{2N}))$.

For the kinetic energy using Proposition 4, we have that f^n satisfies

$$\int_{xv} f^n(t)|v|^2 = \int_{xv} f_0^n |v|^2 + 2t\varepsilon_n \int_{xv} f_0^n \text{ for a.e. } t \in (0, \infty)$$

Again from (36), the right-hand side converges to $\int_{xv} f_0 |v|^2$ as $n \rightarrow \infty$, while the limsup of the left-hand side is lower bounded by $\int_{xv} f(t)|v|^2$, from Fatou's lemma, thus giving the kinetic energy inequality. The same argument can be used to show the moment of inertia inequality.

The conservation of linear momentum is obtained by interpolation. We have, from Cauchy-Schwarz that, for every $k = 1, \dots, N$ and almost every $t \in (0, \infty)$,

$$\int_{xv} |(f^n(t) - f(t))v_k| \leq \left(\int_{xv} |f^n(t) - f(t)| \right)^{1/2} \left(\int_{xv} (f^n(t) + f(t))|v|^2 \right)^{1/2}.$$

The second integral on the right-hand side is bounded uniformly in n , from the inequality of kinetic energy, while the first integral converges to zero for a.e. $t \in (0, \infty)$, since $f^n(t) \rightarrow f(t)$ in $L^1(\mathbb{R}_{x,v}^{2N})$. So, in particular, we have that

$$\int_{xv} f^n(t)v_k \rightarrow \int_{xv} f(t)v_k, \text{ for a.e. } t \in (0, \infty).$$

Part 5: Entropy inequality

In this last part, we prove that the constructed solutions satisfy part (4) of Theorem 1, which corresponds to the entropy inequality. From Proposition 4, the approximated solution satisfies the inequality for n , that is, for almost every $t > 0$,

$$(45) \quad \int_{xv} s^n(t) + \int_0^t \int_{xv} d^n(\tau) d\tau \leq \int_{xv} s^n(0)$$

where s^n and d^n are defined as s and d , but for f^n instead of f and a^n instead of a . Now we want to pass this inequality to the limit.

Let us begin by showing the convergence of the right-hand side. We know that $f_0^n \rightarrow f_0$ almost everywhere, thus $s^n(0) \rightarrow s(0)$ a.e. Since $x \log x - x \leq x \log x + (1-x) \log(1-x)$ for every $x \in [0, 1]$, we have that

$$|s^n(0)| = |f_0^n \log f_0^n + (1 - f_0^n) \log(1 - f_0^n)| \leq |f_0^n \log f_0^n - f_0^n| \leq |f_0^n \log f_0^n| + f_0^n.$$

Thus, since $0 < f_0^n < 1$ and $x \log(\frac{1}{x}) \leq C_0 \sqrt{x}$ for every $x \in (0, 1)$,

$$\begin{aligned} |s^n(0)| &\leq f_0^n \log\left(\frac{1}{f_0^n}\right) + f_0^n \\ &= f_0^n \log\left(\frac{1}{f_0^n}\right) \mathbb{1}_{f_0^n \leq \exp(-|x|-|v|)} + f_0^n \log\left(\frac{1}{f_0^n}\right) \mathbb{1}_{f_0^n \geq \exp(-|x|-|v|)} + f_0^n \\ &\leq C_0 \sqrt{f_0^n} \mathbb{1}_{f_0^n \leq \exp(-|x|-|v|)} + f_0^n (|x| + |v|) + f_0^n \\ &\leq C_0 \exp\left(-\frac{1}{2}(|x| + |v|)\right) + f_0^n (1 + |x| + |v|). \end{aligned}$$

from the convergence of f_0^n and $f_0^n(1 + |x|^2 + |v|^2)$ in $L^1(\mathbb{R}_{x,v}^{2N})$ we have by interpolation that $f_0^n(1 + |x| + |v|)$ also converges in $L^1(\mathbb{R}_{x,v}^{2N})$, and thus by dominated convergence it follows that

$$\int_{xv} s^n(0) \rightarrow \int_{xv} s(0).$$

For the next term, note the same approach as for f_0 works for the function $(x', v) \mapsto f^n(t, x' + tv, v)$, leading

$$|s^n(t, x' + tv, v)| \leq C_0 \exp\left(-\frac{1}{2}(|x'| + |v|)\right) + f^n(t, x' + tv, v)(1 + |x'| + |v|).$$

Then if one considers the variable $x = x' + tv$ we conclude

$$|s^n(t, x, v)| \leq C_0 \exp\left(-\frac{1}{2}(|x - tv| + |v|)\right) + f^n(t, x, v)(1 + |x - tv| + |v|).$$

Since the sequence $f^n(1 + |x - tv|^2 + |v|^2)$ is uniformly bounded in $L^1_{loc}((0, \infty); L^1(\mathbb{R}_{x,v}^{2N}))$, it follows by interpolation that $f^n(1 + |x - tv| + |v|)$ converges in this space, hence by dominated convergence we conclude that s^n should also converge in this space which, passing to a subsequence, implies that

$$\int_{xv} s^n(t) \rightarrow \int_{xv} s(t).$$

for a.e. $t > 0$.

We however cannot directly prove convergence for d^n , because of the derivatives in f . We will instead prove a weak convergence result. Notice d^n can be written as

$$d^n = \left\langle a^n(v - v_*) \sqrt{f^n(1 - f^n)} f_*^n(1 - f_*^n) \left(\frac{\nabla_v f^n}{f^n(1 - f^n)} - \frac{\nabla_{v_*} f_*^n}{f_*^n(1 - f_*^n)} \right), \right. \\ \left. \sqrt{f^n(1 - f^n)} f_*^n(1 - f_*^n) \left(\frac{\nabla_v f^n}{f^n(1 - f^n)} - \frac{\nabla_{v_*} f_*^n}{f_*^n(1 - f_*^n)} \right) \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product.

Since a^n is a positive matrix, let $\sqrt{a^n}$ be the square root of a^n . We can then rewrite d^n as

$$\begin{aligned} & \left| \sqrt{a^n(v - v_*)} \sqrt{f^n(1 - f^n)} f_*^n(1 - f_*^n) \left(\frac{\nabla_v f^n}{f^n(1 - f^n)} - \frac{\nabla_{v_*} f_*^n}{f_*^n(1 - f_*^n)} \right) \right|^2 \\ &= \left| \sqrt{a^n(v - v_*)} \left(\sqrt{f_*^n(1 - f_*^n)} \frac{\nabla_v f^n}{\sqrt{f^n(1 - f^n)}} - \sqrt{f^n(1 - f^n)} \frac{\nabla_{v_*} f_*^n}{\sqrt{f_*^n(1 - f_*^n)}} \right) \right|^2 \\ &= \frac{1}{4} \left| \sqrt{a^n(v - v_*)} \left(\sqrt{f_*^n(1 - f_*^n)} \nabla_v (\arcsin \sqrt{f^n}) - \sqrt{f^n(1 - f^n)} \nabla_{v_*} (\arcsin \sqrt{f_*^n}) \right) \right|^2. \end{aligned}$$

The main idea afterwards is to rewrite the terms inside the square in order to pass to the limit in the sense of distributions. This way, for example, we write

$$(46) \quad \sqrt{a^n(v - v_*)} \sqrt{f_*^n(1 - f_*^n)} \nabla_v (\arcsin \sqrt{f^n}) = \operatorname{div}_v \left(\sqrt{a^n(v - v_*)} \sqrt{f_*^n(1 - f_*^n)} \arcsin \sqrt{f^n} \right) \\ - \operatorname{div}_v \left(\sqrt{a^n(v - v_*)} \right) \sqrt{f_*^n(1 - f_*^n)} \arcsin \sqrt{f^n},$$

and we notate the limit of this sequence as $\sqrt{a(v - v_*)} \sqrt{f_*(1 - f_*)} \nabla_v (\arcsin \sqrt{f})$.

Let $\mathbb{R}_{xv_*}^{3N} = \mathbb{R}_x^N \times \mathbb{R}_v^N \times \mathbb{R}_{v_*}^N$. From uniqueness of the symmetric square root of a positive semi-definite matrix, we have that

$$\sqrt{a^n(z)} = \sqrt{\Gamma^n(z) \psi^n(z) \left(1 + \frac{1}{n|z|^2} \right) P^n(z)},$$

which is uniformly bounded in $L_{loc}^1(\mathbb{R}^N)$. By dominated convergence, one then shows that

$$\sqrt{a^n(v - v_*)} \sqrt{f_*^n(1 - f_*^n)} \arcsin \sqrt{f^n} \rightarrow \sqrt{a(v - v_*)} \sqrt{f_*(1 - f_*)} \arcsin \sqrt{f}$$

in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{xv_*}^{3N})$. Then using (41) we deduce that

$$\operatorname{div}_z \sqrt{a^n(z)} = \sqrt{\Gamma^n(z) \psi^n(z) \left(1 + \frac{1}{n|z|^2} \right) \xi_n(|z|)} \frac{z}{|z|^2},$$

which converges almost everywhere to

$$\sqrt{\Gamma(|z|) \xi(|z|)} \frac{z}{|z|^2} = \operatorname{div}_z \sqrt{a(z)}.$$

Then, using that $\sqrt{1+x} \leq 1 + \sqrt{x}$ for every $x \geq 0$ we conclude

$$\left| \operatorname{div}_z \sqrt{a^n(z)} \right| \leq \sqrt{\Gamma^n(z) \psi^n(z) \xi_n(|z|)} \left(\frac{1}{|z|} + \frac{1}{|z|^2} \right),$$

and therefore $\operatorname{div}_z \sqrt{a^n(z)}$ is dominated by an $L_{loc}^1(\mathbb{R}^N)$ function. Hence from dominated convergence, we have that

$$\operatorname{div}_v \left(\sqrt{a^n(v - v_*)} \right) \sqrt{f_*^n(1 - f_*^n)} \arcsin \sqrt{f^n} \rightarrow \operatorname{div}_v \left(\sqrt{a(v - v_*)} \right) \sqrt{f_*(1 - f_*)} \arcsin \sqrt{f}$$

in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{xvv_*}^{3N})$. Therefore the right hand side of (46) converges in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{xvv_*}^{3N})$ to

$$\operatorname{div}_v \left(\sqrt{a(v - v_*)} \sqrt{f_*(1 - f_*)} \arcsin \sqrt{f} \right) - \operatorname{div}_v \left(\sqrt{a(v - v_*)} \right) \sqrt{f_*(1 - f_*)} \arcsin \sqrt{f}$$

which we will notate, by correspondence with the smooth case, $\sqrt{a(v - v_*)} \sqrt{f_*(1 - f_*)} \nabla_v (\arcsin \sqrt{f})$. Exchanging v and v_* we show similarly that

$$\sqrt{a^n(v - v_*)} \sqrt{f^n(1 - f^n)} \nabla_{v_*} (\arcsin \sqrt{f^n}) \rightarrow \sqrt{a(v - v_*)} \sqrt{f(1 - f)} \nabla_{v_*} (\arcsin \sqrt{f_*})$$

in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{xvv_*}^{3N})$, where the distribution on the left is a notation, defined analogously to the previous case. Subtracting both convergences, we conclude that

$$\begin{aligned} \sqrt{a^n(v - v_*)} \left(\sqrt{f_*^n(1 - f_*^n)} \nabla_v (\arcsin \sqrt{f^n}) - \sqrt{f^n(1 - f^n)} \nabla_{v_*} (\arcsin \sqrt{f_*^n}) \right) \rightarrow \\ \sqrt{a(v - v_*)} \left(\sqrt{f_*(1 - f_*)} \nabla_v (\arcsin \sqrt{f}) - \sqrt{f(1 - f)} \nabla_{v_*} (\arcsin \sqrt{f_*}) \right) \end{aligned}$$

in $\mathcal{D}'((0, \infty) \times \mathbb{R}_{xvv_*}^{3N})$. Notice that inequality (45) implies that the left-hand side of this convergence is in fact uniformly bounded in $L^2((0, \infty) \times \mathbb{R}_{xvv_*}^{3N})$, hence the convergence above actually holds weakly in this space, and the distribution in the right-hand side can be identified to an L^2 function.

Then, by the lower semi-continuity of the norm under weak convergence, it follows that, for almost every $t > 0$, we have

$$\int_0^t \int_{x,v,v_*} d(\tau) \, d\tau \leq \liminf_{n \rightarrow \infty} \int_0^t \int_{x,v,v_*} d^n(\tau) \, d\tau,$$

where we have defined

$$d = \frac{1}{4} \left| \sqrt{a(v - v_*)} \left(\sqrt{f_*(1 - f_*)} \nabla_v (\arcsin \sqrt{f}) - \sqrt{f(1 - f)} \nabla_{v_*} (\arcsin \sqrt{f_*}) \right) \right|^2$$

Thus passing the \liminf in (45), the entropy inequality follows. \square

REFERENCES

- [1] AGOSHKOV, V. I. Spaces of functions with differential-difference characteristics and the smoothness of solutions of the transport equation. *Dokl. Akad. Nauk SSSR* 276, 6 (1984), 1289–1293.
- [2] ALEXANDRE, R. On some related non homogeneous 3D Boltzmann models in the non cutoff case. *Journal of Mathematics of Kyoto University* 40, 3 (2000), 493 – 524.
- [3] ALONSO, R., BAGLAND, V., DESVILLETES, L., AND LODS, B. About the Landau-Fermi-Dirac Equation With Moderately Soft Potentials. *Archive for Rational Mechanics and Analysis* 244, 3 (June 2022), 779–875.
- [4] BAGLAND, V. Well-posedness for the spatially homogeneous landau–fermi–dirac equation for hard potentials. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 134, 3 (2004), 415–447.
- [5] BOUCHUT, F., GOLSE, F., AND PULVIRENTI, M. *Kinetic equations and asymptotic theory*. Series in Applied Mathematics. Elsevier, 2000.
- [6] DESVILLETES, L., AND VILLANI, C. On the spatially homogeneous landau equation for hard potentials part i : existence, uniqueness and smoothness. *Communications in Partial Differential Equations* 25, 1-2 (2000), 179–259.
- [7] DiPERNA, R. J., AND LIONS, P. L. On the cauchy problem for boltzmann equations: Global existence and weak stability. *Annals of Mathematics* 130, 2 (1989), 321–366.
- [8] DOLBEAULT, J. Kinetic models and quantum effects: A modified Boltzmann equation for Fermi-Dirac particles. *Archive for Rational Mechanics and Analysis* 127, 2 (June 1994), 101–131.
- [9] GOLDING, W., GUALDANI, M. P., AND ZAMPONI, N. Existence of smooth solutions to the Landau–Fermi–Dirac equation with Coulomb potential. *Communications in Mathematical Sciences* 20, 8 (2022), 2315–2365.
- [10] GOLSE, F., LIONS, P.-L., PERTHAME, B., AND SENTIS, R. Regularity of the moments of the solution of a transport equation. *Journal of Functional Analysis* 76, 1 (1988), 110–125.
- [11] HÖRMANDER, L. Hypoelliptic second order differential equations. *Acta Mathematica* 119, none (1967), 147 – 171.
- [12] IL’IN, A. M. On a class of ultraparabolic equations. *Sov. Math., Dokl.* 5 (1964), 1673–1676.
- [13] KOLMOGOROFF, A. Zufällige bewegungen (zur theorie der brownischen bewegung). *Annals of Mathematics* 35, 1 (1934), 116–117.

- [14] LIONS, P. L. On Boltzmann and Landau Equations. *Philosophical Transactions: Physical Sciences and Engineering* 346, 1679 (1994), 191–204.
- [15] VILLANI, C. On the Cauchy problem for Landau equation: sequential stability, global existence. *Advances in Differential Equations* 1, 5 (1996), 793 – 816.
- [16] WEBER, M. The fundamental solution of a degenerate partial differential equation of parabolic type. *Transactions of the American Mathematical Society* 71, 1 (1951), 24–37.

CENTRE DE MATHÉMATIQUES LAURENT SCHWARTZ, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE
Email address: paulo-elpidio.alves-sampaio@polytechnique.edu