Lecture 2

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1 Spectral theorem and diagonalization

Theorem 1. *Let A be a real, symmetric matrix (or, in the complex case, a self-adjoint matrix). Then there exist D a real diagonal matrix and P an orthogonal (unitary) matrix such that $A = PDP^*$.*

Theorem 2. *If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.*

1.1 Powers of matrices

The idea here is to take advantage of the fact that if a matrix A can be written as PDP^{-1} then its powers are given by $A^k = PD^kP^{-1}$, the powers of a diagonal matrices being easy to compute.

Problem 1. *Let T be an $N \times N$ real symmetric matrix. Show that*

$$\lim_{n\to\infty} T^n = 0$$

if and only if all the eigenvalues of T have absolute value less than 1.

Problem 2 (Berkeley 1990)**.** *Let A be a real symmetric $n \times n$ matrix that is positive definite. Let $y \in \mathbb{R}^N$, $y \neq 0$. Prove that the limit*

$$\lim_{m\to\infty} \frac{y^T A^{m+1} y}{y^T A^m y}$$

exists and is an eigenvalue of A.

1.2 Linear recurrences

A linear recurrence with constant coefficients

$$A_{n+2} = aA_{n+1} + bA_n$$

can be transformed into a linear system by introducing the artificial sequence $B_{n+2} = A_{n+1}$, giving

$$\begin{cases} A_{n+2} = aA_{n+1} + bB_{n+1} \\ B_{n+2} = A_{n+1} \end{cases} \implies \begin{pmatrix} A_{n+2} \\ B_{n+2} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix}$$

Problem 3. *Show that if $a^2 + 4b > 0$ there exist constants α, β such that the solution to the linear recurrence above is given by*

$$A_n = \alpha\lambda_1^n + \beta\lambda_2^n,$$

where λ_1 and λ_2 are the eigenvalues of some matrix.

Problem 4. *Prove the Binet formula for the Fibonacci sequence $0, 1, 1, 2, \ldots$:*

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Problem 5. *Find all sequences satisfying $f_{n+2} = 5f_{n+1} + 6f_n + 2^n$*

Problem 6 (Putnam 2018)**.** *Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_nx_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that, if $x_N = 0$ for some N, then the sequence is periodic. (Hint: Show that we must have $|a| \leq 1$ and show that $x_n = \cos(y_nb)$).*

1.3 Courant-Fischer min-max theorem

Theorem 3. *Let A be a real symmetric matrix and note $\lambda_1, \lambda_2, \ldots, \lambda_n$ its eigenvalues. Then we have*

$$\lambda_k = \min_{\dim S = n-k+1} \left(\max_{x \in S} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \right)$$

In particular we have that

$$\lambda_{min} \leq \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \leq \lambda_{max},$$

where the left and right inequalities are obtained exactly when x is an eigenvector corresponding to λ_{min} and λ_{max}, respectively (prove it as an exercise).

Problem 7 (OBM 2004)**.** *Let X be a real invertible $n \times n$ matrix and X^T its transposed. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ the eigenvalues of X^TX. We define the norm of X by $\|X\| = \sqrt{\lambda_1}$ and the dilation factor of A by $d(X) = \sqrt{\frac{\lambda_1}{\lambda_2}}$. Show that, for any A and B invertible, $d(AB) \geq \frac{\|AB\|}{\|A\|\|B\|} d(B)$.*

Problem 8. *Prove the eigenvalue stability inequality $|\lambda_i(A+B) - \lambda_i(A)| \leq \|B\|_{op}$, where we define the norm $\|M\|_{op} = \sup_{|x|=1} |Mx|$.*

Problem 9 (Berkeley 1992)**.** *Let A be a real symmetric $n \times n$ matrix with non-negative entries. Prove that A has an eigenvector with non-negative entries.*

2 Polar Decomposition

Any complex number $z = a + bi$ can be put in the polar form $z = re^{i\varphi}$, where $r \geq 0$ and $|e^{i\varphi}| = 1$. Here we will see the generalization of this fact to square matrices, where a symmetric matrix takes the role of the r and a unitary matrix acts as the phase factor.

Theorem 4. *Any square matrix A over $\mathbb{R}$ (or $\mathbb{C}$) can be represented in the form $A = SU$, where S is a symmetric (Hermitian) non-negative definite matrix and U is an orthogonal (unitary) matrix. If A is invertible such a representation is unique.*

Problem 10. *Prove that S is always uniquely defined, $S = \sqrt{AA^*}$.*

Problem 11. *Prove that any square matrix can be decomposed as $A = U'S'$, where S' is a symmetric non-negative definite matrix and U' is an orthogonal (unitary) matrix. (Note that we don't have (in general) $U' = U$ and $S' = S$.)*

Problem 12. *Prove that if A is invertible and $A = S_1U_1 = U_2S_2$, where S_i are symmetric and U_i are unitary, then $U_1 = U_2$.*

Problem 13. *Prove that if the polar decomposition of a square matrix A is unique then A must be invertible.*

Problem 14. *Prove that if U is a unitary matrix and $S \geq 0$, then $|tr(US)| \leq trS$ and if A is invertible, then $U = e^{i\varphi}I$, for some $\varphi \in \mathbb{R}$.*

Problem 15. *Let $A = SU$ be the polar decomposition of A and W a unitary matrix. Then $\|A - U\|_2 \leq \|A - W\|_2$ and if A is invertible, then the equality is only attained for $W = U$ (Use the result from Problem 14 above). We recall that the L^2 euclidean norm of a matrix A is $\|A\|_2 = \sqrt{tr(A^*A)}$.*

Problem 16. *Prove that if A is a normal operator (that is, $AA^* = A^*A$) and $A = SU$ is its polar decomposition then $SU = US$.*

2.1 Singular value decomposition

The matrix S in the polar decomposition, being itself a symmetric matrix, can be further decomposed as $S = PDP^*$ using the spectral theorem. Since $S \geq 0$, it follows that the eigenvalues in the diagonal matrix D are non-negative. This way, there exists two orthogonal (unitary) matrices P and Q such that A can be "almost diagonalized" as $A = PDQ$. The elements in the diagonal D are called the *singular values* of A, and this is called the *singular value decomposition*.

Problem 17. *If A is an $n \times n$ normal matrix, show that the singular values $\sigma_1(A), \ldots, \sigma_n(A)$ of A are the absolute values of its eigenvalues: $|\lambda_1(A)|, \ldots, |\lambda_n(A)|$.*

Problem 18. *Prove the Courant-Fischer min-max formula for singular values:*

$$\sigma_i(A) = \inf_{dim(S)=n-i+1} \left(\sup_{x \in S} \frac{|Av|}{|v|} \right)$$

for all $1 \leq i \leq p$, where the supremum ranges over all subspaces of $\mathbb{C}^n$ of dimension i.