Lecture 2

Paulo Sampaio

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1 Spectral theorem and diagonalization

Theorem 1. *Let A be a real, symmetric matrix (or, in the complex case, a self-adjoint matrix). Then there exist D a real diagonal matrix and P an orthogonal (unitary) matrix such that $A = PDP^*$.*

Theorem 2. *If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.*

1.1 Powers of matrices

The idea here is to take advantage of the fact that if a matrix A can be written as PDP^{-1} then its powers are given by $A^k = PD^kP^{-1}$, the powers of a diagonal matrices being easy to compute.

Problem 1. *Let T be an $N \times N$ real symmetric matrix. Show that*

$$\lim_{n\to\infty} T^n = 0$$

if and only if all the eigenvalues of T have absolute value less than 1.

Solution: ($\Longleftarrow$) Diagonalizing T with the spectral theorem leads $T = PDP^* \implies T^n = PD^nP^*$. Notating by λ_i the eigenvalues of T, we have

$$\begin{aligned} T^n_{ij} &= \sum_{k,l=1}^{N} P_{ik}(D^n)_{kl}P^*_{lj} \\ &= \sum_{k=1}^{N} P_{ik}(D^n)_{kk}P^*_{kj} \\ &= \sum_{k=1}^{N} P_{ik}P^*_{kj}\lambda^n_k \end{aligned}$$

This way, if $|\lambda_k| < 1$, then $P_{ik}P^*_{kj}\lambda^n_k \to 0$ and thus $T^n_{ij} \to 0$.

($\implies$) We invert the diagonalization as $D^n = P^*T^nP$, where we have used that P is an unitary matrix ($PP^* = P^*P = I$). This implies that

$$\lambda^n_i = D_i i^n = \sum_{k,l=1}^{N} P^*_{ik}T^n_{kl}P_{li}.$$

Thus if $\lim_{n\to\infty} T^n_{kl} = 0$, then $\lim_{n\to\infty} \lambda^n_i = 0$, which implies $|\lambda_i| < 1$.

Problem 2 (Berkeley 1990)**.** *Let A be a real symmetric $n \times n$ matrix that is positive definite. Let $y \in \mathbb{R}^N$, $y \neq 0$. Prove that the limit*

$$\lim_{m\to\infty} \frac{y^TA^{m+1}y}{y^TA^my}$$

exists and is an eigenvalue of A.

Solution: Notice that $y^TA^{m+1}y = \langle y, A^{m+1}y\rangle$ and thus the limit we want to evaluate can be rewritten as

$$\lim_{m\to\infty} \frac{\langle y, A^{m+1}y\rangle}{\langle y, A^my\rangle}.$$

By the spectral theorem there exist a diagonal matrix D and a unitary matrix P such that $A = P^*DP$. Using that $\langle Ax, x\rangle = \langle x, A^*x\rangle$, this quotient can be rewritten as

$$\frac{\langle y, P^*D^{m+1}Py\rangle}{\langle y, P^*D^mPy\rangle} = \frac{\langle Py, D^{m+1}Py\rangle}{\langle Py, D^mPy\rangle} = \frac{\langle z, D^{m+1}z\rangle}{\langle z, D^mz\rangle},$$

where we define $z = Py$. Expanding the inner products we find that this equals

$$\frac{\sum_{i=1}^N \lambda_i^{m+1} z_i^2}{\sum_{i=1}^N \lambda_i^m z_i^2},$$

where the λ_is are the eigenvalues of the matrix A. Let λ_M be the greatest eigenvalue.

The numerator behaves as λ_M^{m+1} and the denominator as λ_M^m. Indeed, the above fraction equals

$$\frac{\lambda_M^{m+1}\sum_{i=1}^N (\frac{\lambda_i}{\lambda_M})^{m+1} z_i^2}{\lambda_M^m\sum_{i=1}^N (\frac{\lambda_i}{\lambda_M})^m z_i^2} = \lambda_M \frac{\sum_{i=1}^N (\frac{\lambda_i}{\lambda_M})^{m+1} z_i^2}{\sum_{i=1}^N (\frac{\lambda_i}{\lambda_M})^m z_i^2} = \lambda_M \frac{z_M^2 + \sum_{i\neq M} (\frac{\lambda_i}{\lambda_M})^{m+1} z_i^2}{z_M^2 + \sum_{i\neq M} (\frac{\lambda_i}{\lambda_M})^m z_i^2}$$

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1.2 Linear recurrences

A linear recurrence with constant coefficients

$$A_{n+2} = aA_{n+1} + bA_n$$

can be transformed into a linear system by introducing the artificial sequence $B_{n+2} = A_{n+1}$, giving

$$\begin{cases} A_{n+2} = aA_{n+1} + bB_{n+1} \\ B_{n+2} = A_{n+1} \end{cases} \implies \begin{pmatrix} A_{n+2} \\ B_{n+2} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix}$$

Problem 3. *Show that if $a^2 + 4b > 0$ there exist constants α, β such that the solution to the linear recurrence above is given by*

$$A_n = \alpha\lambda_1^n + \beta\lambda_2^n,$$

where λ_1 and λ_2 are the eigenvalues of some matrix.

Problem 4. *Prove the Binet formula for the Fibonacci sequence $0, 1, 1, 2, \dots$:*

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Solution: Here we come exactly to the case of the previous question. The matrix of the system is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

with characteristic polynomial $\lambda^2 - \lambda - 1 = 0$, we find that the eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

It remains just to find the values of the constants α and β, which can be done by substituting the initial cases $F_0 = 0$ and $F_1 = 1$.

Problem 5. *Find all sequences satisfying $f_{n+2} = 5f_{n+1} + 6f_n + 2^n$*

Solution: To get rid of the 2^n term, note that the recurrence relation for the $n+1$ term of the sequence is written as

$$f_{n+1} = 5f_n + 6f_{n-1} + 2^{n-1} \implies 2f_{n+1} = 52f_n + 62f_{n-1} + 2^n$$

Thus, subtracting this one from the original equation leads

$$f_{n+2} - 2f_{n+1} = 5(f_{n+1} - 2f_n) + 6(f_n - 2f_{n-1}).$$

Define the auxiliary recurrent sequence $g_{n+1} = f_{n+1} - 2f_n$ for every $n \geq 0$. The above equation is then stated as

$$g_{n+2} = 5g_{n+1} + 6g_n.$$

Since $5^2 + 4 * 6 > 0$, we have by Problem 3 that the general solution to such a recurrence is given by

$$g_n = \alpha\lambda_1^n + \beta\lambda_2^n,$$

where the λ_i are the eigenvalues of the matrix $\begin{pmatrix} 5 & 6 \\ 1 & 0 \end{pmatrix}$, namely, 6 and -1.

By the definition of g, we have that

$$f_{n+1} - 2f_n = \alpha 6^n + \beta(-1)^n, \text{ for every } n \geq 0.$$

This recurrence can be solved by summation. Indeed, writing all the recurrences decreasing order from n to 0:

$$\begin{aligned} f_{n+1} - 2f_n &= \alpha 6^n + \beta(-1)^n \\ f_n - 2f_{n-1} &= \alpha 6^{n-1} + \beta(-1)^{n-1} \\ f_{n-1} - 2f_{n-2} &= \alpha 6^{n-2} + \beta(-1)^{n-2} \\ &\vdots \\ f_1 - 2f_0 &= \alpha + \beta \end{aligned}$$

The idea is to cancel the $-f_n$ with the f_n on the equation below, then the f_{n-1} with the f_{n-2} on the equation below and so on. To do this, multiply the second equation by 2, the third one by 2^2 and so on such that the last equation will be multiplied by 2^n, and we have

$$\begin{aligned} f_{n+1} - 2f_n &= \alpha 6^n + \beta(-1)^n \\ 2f_n - 2^2 f_{n-1} &= 2\alpha 6^{n-1} + 2\beta(-1)^{n-1} \\ 2^2 f_{n-1} - 2^3 f_{n-2} &= 2^2\alpha 6^{n-2} + 2^2\beta(-1)^{n-2} \\ &\vdots \\ 2^n f_1 - 2^{n+1} f_0 &= 2^n\alpha + 2^n\beta \end{aligned}$$

Summing everything we have the desired cancellations and it follows that

$$f_{n+1} - 2^{n+1}f_0 = \alpha(6^n + 2\cdot 6^{n-1} + 2^2\cdot 6^{n-2} + \cdots + 2^n) + \beta((-1)^n + 2\cdot(-1)^{n-1} + 2^2\cdot(-1)^{n-2} + \cdots + 2^n)$$

or

$$f_{n+1} - 2^{n+1}f_0 = \alpha\sum_{i=0}^{n} 2^i 6^{n-i} + \beta\sum_{i=0}^{n} 2^i(-1)^{n-i}$$

Problem 6 (Putnam 2018)**.** *Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_nx_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that, if $x_N = 0$ for some N, then the sequence is periodic. (Hint: Show that we must have $|a| \leq 1$ and show that $x_n = \cos(y_nb)$).*

Solution: Let us first motivate the hint. Calculating the next two elements of this sequence we have that

$$\begin{aligned} x_3 &= 2a^2 - 1 \\ x_4 &= 4a^3 - 3a \end{aligned}$$

and these formulas look a lot like the formulas for $\cos(2x)$ and $\cos(3x)$ as a function of $\cos(x)$. Thus it makes sense to search for solutions of the form $\cos(y_nb)$, for some sequence y_n.

Now, let us justify the substitution. Suppose $|a| > 1$. We can show by induction that $|x_{n+1}| \geq |x_n|$. Thus in this case we cannot have $x_n = 0$ for some n, meaning we must have $|a| \leq 1$.

Let b be such that $a = \cos(b)$. This way, we have $y_1 = y_2 = 1$, $y_3 = 2$, $y_4 = 3$. Constructing the rest of y_n by induction, if $x_n = \cos(y_nb)$, $x_{n-1} = \cos(y_{n-1}b)$ and $x_{n-2} = \cos(y_{n-2}b)$ we have

$$\begin{aligned} x_{n+1} &= 2\cos(y_nb)\cos(y_{n-1}b) - \cos(y_{n-2}b) \\ &= \cos((y_n + y_{n-1})b) + \cos((y_n - y_{n-1})b) - \cos(y_{n-2}b). \end{aligned}$$

Now we describe some heuristics to construct the y_n sequence. Intuitively, since we want $x_{n+1} = \cos(y_{n+1}b)$, one of the cos must cancel the $-\cos(y_{n-2}b)$. We have two options: either $y_n + y_{n-1} = y_{n-2}$ or $y_n - y_{n-1} = y_{n-2}$.

The first recurrence doesn't agree with the first values of y_n that we calculated earlier. Thus we impose that $y_n - y_{n-1} = y_{n-2}$, and comparing with the first values we have calculated, we conclude that y_n is in fact the Fibonacci sequence! This way, we have that $x_{n+1} = \cos(y_{n+1}b)$, as desired.

Now we prove the periodicity. Indeed, if $x_N = 0$, then this implies $y_N b = (k + \frac{1}{2})\pi$ for some k. Thus $x_n = \cos(\frac{y_n}{y_N}(k + \frac{1}{2})\pi)$.

1.3 Courant-Fischer min-max theorem

Theorem 3. *Let A be a real symmetric matrix and note $\lambda_1, \lambda_2, \ldots, \lambda_n$ its eigenvalues. Then we have*

$$\lambda_k = \min_{\dim S = n-k+1} \left(\max_{x \in S} \frac{\langle Ax, x \rangle}{\langle x, x \rangle} \right)$$

In particular we have that

$$\lambda_{min} \leq \frac{\langle Ax, x \rangle}{\langle x, x \rangle} \leq \lambda_{max},$$

where the left and right inequalities are obtained exactly when x is an eigenvector corresponding to λ_{min} and λ_{max}, respectively (prove it as an exercise).

Problem 7 (OBM 2004)**.** *Let X be a real invertible $n \times n$ matrix and X^T its transposed. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ the eigenvalues of $X^T X$. We define the norm of X by $\|X\| = \sqrt{\lambda_1}$ and the dilation factor of A by $d(X) = \sqrt{\frac{\lambda_1}{\lambda_2}}$. Show that, for any A and B invertible, $d(AB) \geq \frac{\|AB\|}{\|A\|\|B\|} d(B)$.*

Solution: For a matrix X with real eigenvalues, we will notate $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ all of its eigenvalues. The norm defined in the statement of the problem then is written as $\|X\| = \sqrt{\lambda_1(X^T X)}$. Also, the dilation factor is written as $d(X) = \sqrt{\frac{\lambda_1(X^T X)}{\lambda_2(X^T X)}}$.

Using these definitions,

$$\begin{aligned}
d(AB) \geq \frac{\|AB\|}{\|A\|\|B\|} d(B) &\Leftrightarrow d(AB)^2 \geq \frac{\|AB\|^2}{\|A\|^2\|B\|^2} d(B)^2 \\
&\Leftrightarrow \frac{\lambda_1((AB)^T(AB))}{\lambda_2((AB)^T(AB))} \geq \frac{\lambda_1((AB)^T(AB))}{\lambda_1(A^T A)\lambda_1(B^T B)} \frac{\lambda_1(B^T B)}{\lambda_2(B^T B)} \\
&\Leftrightarrow \frac{1}{\lambda_2((AB)^T(AB))} \geq \frac{1}{\lambda_1(A^T A)\lambda_2(B^T B)} \\
&\Leftrightarrow \lambda_2((AB)^T(AB)) \leq \lambda_1(A^T A)\lambda_2(B^T B)
\end{aligned}$$

and we just have to prove this last inequality.

By Courant-Fischer,

$$\begin{aligned}
\lambda_2((AB)^T(AB)) &= \min_{\dim S = n-1} \left(\max_{x \in S} \frac{\langle (AB)^T ABx, x \rangle}{\langle x, x \rangle} \right) \\
&= \min_{\dim S = n-1} \left(\max_{x \in S} \frac{\langle ABx, ABx \rangle}{\langle x, x \rangle} \right) \\
&= \min_{\dim S = n-1} \left(\max_{x \in S} \frac{\|Ax\|_2^2}{\|x\|_2^2} \right)
\end{aligned}$$

But

$$\max_{x \in S} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \max_{x \in S} \left(\frac{\|ABx\|_2^2}{\|Bx\|_2^2} \frac{\|Bx\|_2^2}{\|x\|_2^2} \right) \leq \max_{x \in S} \frac{\|ABx\|_2^2}{\|Bx\|_2^2} \cdot \max_{x \in S} \frac{\|Bx\|_2^2}{\|x\|_2^2}$$

Note that since B is invertible, a reparametrization in the max implies

$$\max_{x \in S} \frac{\|ABx\|_2^2}{\|Bx\|_2^2} \leq \max_{x \in \mathbb{R}^n} \frac{\|ABx\|_2^2}{\|Bx\|_2^2} = \max_{x \in \mathbb{R}^n} \frac{\|Ax\|_2^2}{\|x\|_2^2}$$

(note this last equality wouldn't be possible if the max was taken only over S). Hence we have

$$\max_{x \in S} \frac{\|Ax\|_2^2}{\|x\|_2^2} \leq \max_{x \in S} \frac{\|Ax\|_2^2}{\|x\|_2^2} \cdot \max_{x \in S} \frac{\|Bx\|_2^2}{\|x\|_2^2}.$$

In a similar way as before we have, by Courant-Fischer, that

$$\lambda_1(A^T A) = \max_{x \in \mathbb{R}^n} \frac{\|Ax\|_2^2}{\|x\|_2^2} \text{ and } \lambda_2(B^T B) = \min_{\dim S = n-1} \max_{x \in S} \frac{\|Bx\|_2^2}{\|x\|_2^2},$$

and this implies

$$\max_{x \in S} \frac{\|Ax\|_2^2}{\|x\|_2^2} \leq \lambda_1(A^T A) \cdot \max_{x \in S} \frac{\|Bx\|_2^2}{\|x\|_2^2}.$$

Taking the minimum over all subspaces S with dimension $n-1$ leads, by Courant-Fischer,

$$\lambda_2((AB)^T(AB)) \leq \lambda_1(A^T A)\lambda_2(B^T B)$$

which is equivalent to the inequality that we wanted.

Problem 8. *Prove the eigenvalue stability inequality* $|\lambda_i(A+B) - \lambda_i(A)| \leq \|B\|_{op}$*, where we define the norm* $\|M\|_{op} = \sup_{|x|=1} |Mx|$.

Solution: We have that

$$\max_{x \in S} \frac{\langle (A+B)x, x\rangle}{\langle x, x\rangle} \leq \max_{x \in S} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} + \max_{x \in S} \frac{\langle Bx, x\rangle}{\langle x, x\rangle}$$

But, by Cauchy-Schwarz inequality, we have that

$$\frac{\langle Bx, x\rangle}{\langle x, x\rangle} \leq \frac{|Bx||x|}{|x|^2} = \frac{|Bx|}{|x|} = \left| B\left(\frac{x}{|x|}\right)\right| \leq \max_{|x|=1} |Bx| = \|B\|_{op}$$

Thus,

$$\max_{x \in S} \frac{\langle (A+B)x, x\rangle}{\langle x, x\rangle} \leq \max_{x \in S} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} + \|B\|_{op}$$

and taking the minimum over all subspaces S with dimension $n-i+1$ leads, by Courant-Fischer,

$$\lambda_i(A+B) \leq \lambda_i(A) + \|B\|_{op}$$

For the other inequality, it suffices do choose $A+B$ in the place of A and $-B$ in place of B. We then have

$$\lambda_i((A+B)+(-B)) \leq \lambda_i(A+B) + \|-B\|_{op} \implies \lambda_i(A+B) \geq \lambda_i(A) - \|B\|_{op}$$

and the inequality is shown.

Problem 9 (Berkeley 1992). *Let A be a real symmetric $n \times n$ matrix with non-negative entries. Prove that A has an eigenvector with non-negative entries.*

Solution: Let λ_{max} be the maximum eigenvalue for this matrix and let $v = (v_1, v_2, \ldots, v_n)$ be such that $Av = \lambda_{max} v$. Consider the vector $|v| \equiv (|v_1|, |v_2|, \ldots, |v_n|)$. Since A has non-negative entries, it follows that

$$\frac{\langle A|v|, |v|\rangle}{\langle |v|, |v|\rangle} = \frac{\sum_{ij=1}^n A_{ij}|v_i||v_j|}{\||v|\|_2^2} \geq \frac{\sum_{ij=1}^n A_{ij} v_i v_j}{\|v\|_2^2} = \frac{\langle Av, v\rangle}{\langle v, v\rangle}.$$

However, as $Av = \lambda_{max} v$, we have the saturation in Courant-Fischer:

$$\frac{\langle Av, v\rangle}{\langle v, v\rangle}$$

thus we have

$$\frac{\langle A|v|, |v|\rangle}{\langle |v|, |v|\rangle} \geq \lambda_{max},$$

but since the Rayleigh coefficient is always less than λ_{max}, we have equality in the inequality above. This in turn implies that $|v|$ must be an eigenvector associated with the eigenvalue λ_{max}.

2 Polar Decomposition

Any complex number $z = a + bi$ can be put in the polar form $z = re^{i\varphi}$, where $r \geq 0$ and $|e^{i\varphi}| = 1$. Here we will see the generalization of this fact to square matrices, where a symmetric matrix takes the role of the r and a unitary matrix acts as the phase factor.

Theorem 4. *Any square matrix A over $\mathbb{R}$ (or $\mathbb{C}$) can be represented in the form $A = SU$, where S is a symmetric (Hermitian) non-negative definite matrix and U is an orthogonal (unitary) matrix. If A is invertible such a representation is unique.*

Problem 10. *Prove that S is always uniquely defined, $S = \sqrt{AA^*}$.*

Solution: If $A = SU$ then we have that $A^* = U^*S^* = U^*S$. Thus $AA^* = SUU^*S = S^2$, since U is unitary. Now it suffices to take the matrix square root on both sides, which is a well defined operation in this case, since $S \geq 0$.

Problem 11. *Prove that any square matrix can be decomposed as $A = U'S'$, where S' is a symmetric non-negative definite matrix and U' is an orthogonal (unitary) matrix. (Note that we don't have (in general) $U' = U$ and $S' = S$.)*

Solution: Take the polar decomposition of the conjugate matrix $A^* = SV$, where $S \geq 0$ is a symmetric and V is unitary. Now, taking the hermitian conjugation on both sides of the equation leads $A = V^*S^* = V^*S$ and it suffices to take $U = V^*$.

Problem 12. *Prove that if A is invertible and $A = S_1U_1 = U_2S_2$, where S_i are symmetric and U_i are unitary, then $U_1 = U_2$.*

Solution: "Insert and remove" an unitary matrix on the right-hand side of the first equation: $A = S_1U_1 = U_1U_1^*S_1U_1 = U_1(U_1^*S_1U_1)$. This way we have $U_2S_2 = U_1(U_1^*S_1U_1)$.

Since A is invertible and $U_1^*S_1U_1 \geq 0$, it follows by uniqueness of the polar decomposition that

$$U_1 = U_2 \text{ and } S_2 = U_1^*S_1U_1.$$

Problem 13. *Prove that if the polar decomposition of a square matrix A is unique then A must be invertible.*

Solution: Let us prove the converse statement, that is, if A is not invertible then it's polar decomposition cannot be unique. If A is not invertible, this implies there exists a v such that $Av = 0$, that is, there's a direction v that is totally killed by A.

What this implies is that any linear transformation that affects only v will not change A. This way, let H be the reflection on the direction v, that is, a linear transformation such that $Hv = -v$ and $Hw = w$ for any $w \in v^{\perp}$. As every reflection, the map H is an unitary application, for it is an isometry $\|Hv\| = \|v\|$.

Now we show rigorously that applying H won't change A. Every x can be decomposed as $x = \lambda v + w$, where v and w are as above. We have $AHx = \lambda AHv + AHw = -\lambda Av + Aw = Aw = \lambda Av + Aw = Ax$.

Take $A = SU$ a polar decomposition of A. We have that $A = AH = SUH = S(UH)$, and thus this decomposition is not unique.

Problem 14. *Prove that if U is a unitary matrix and $S \geq 0$, then $|tr(US)| \leq trS$ and if A is invertible, then the equality holds only when $U = e^{i\varphi}I$, for some $\varphi \in \mathbb{R}$.*

Solution: Let $S = VDV^*$, with $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ and V an unitary matrix.

$$tr(US) = tr(UV^*DV) = tr(VUV^*D) = tr(WD),$$

where we have defined $W = VUV^*$, an unitary matrix. This way

$$tr(US) = \sum_{i=1}^{n}(WD)_{ii} = \sum_{i,j=1}^{n} W_{ij}D_{ji} = \sum_{i=1}^{n} W_{ii}\lambda_i$$

$$|tr(US)| \leq \sum_{i=1}^{n} |w_{ii}|\lambda_i \leq \sum_{i=1}^{n} \lambda_i = tr(S)$$

and the desired inequality is proven.

For the equality case, since A is invertible, we have that $S > 0$ and thus $\lambda_i > 0$. Then,

$$\left|\sum_{i=1}^{n} w_{ii}\lambda_i\right| = \sum_{i=1}^{n} |w_{ii}|\lambda_i = \sum_{i=1}^{n} \lambda_i.$$

The last equality is an equality case for the convex combination $|w_{11}|\lambda_1 + \cdots + |w_{nn}|\lambda_n \leq \lambda_1 + \cdots + \lambda_n$ (remember that since W is unitary, we have $|w_i i| \leq 1$), which can only happen if $|w_{ii}| = 1$ for every i. Since W is an unitary matrix, each column of W has to be a norm 1 vector, thus we deduce that W must be a diagonal matrix.

Also, the first equality above is the equality case for the triangular inequality, which implies there exist $\alpha_i \geq 0$ such that $w_{ii}\lambda_i = \alpha_i w_{11}\lambda_1\ \forall i$ or simply that there exists a $\beta_i \geq 0$ such that $w_i i = \beta_i w_{11}\ \forall i$.

Taking the norms we deduce that $\beta_i = 1$ and thus the w_{ii} are all equal. Let $\varphi \in \mathbb{R}$ be such that $w_{11} = e^{i\varphi}$. We have that $W = e^{i\varphi}I$, and thus $U = e^{i\varphi}I$, as desired.

Problem 15. *Let $A = SU$ be the polar decomposition of A and W a unitary matrix. Then $\|A - U\|_2 \leq \|A - W\|_2$ and if A is invertible, then the equality is only attained for $W = U$ (Use the result from Problem 14 above). We recall that the L^2 euclidean norm of a matrix A is $\|A\|_2 = \sqrt{tr(A^*A)}$.*

Solution: Note that if U is a unitary matrix, then

$$\|AU\|_2^2 = tr((U^*A^*A)U) = tr(U(U^*A^*A)) = tr(A^*A) = \|A\|_2^2,$$

that is, unitary matrices preserve this norm.

If $A = SU$ is the polar decomposition of A, then

$$\|A - W\|_2 = \|SU - W\|_2 = \|SUU^* - WU^*\|_2 = \|S - WU^*\|_2 = \|S - V\|_2$$

where we define $V = WU^*$, a unitary matrix.

On the other hand, by the definition of the norm,

$$\|S - V\|_2^2 = tr((S - V)(S - V^*)) = tr(S^2 - SV^* - VS + VV^*) = tr(S^2) - tr(SV^* + VS) + tr(I).$$

Using the result from problem 18, we have that $|tr(SV)| \leq trS$ (and $|tr(V^*S)| \leq trS$ by taking the hermitian adjoint), thus

$$\|S - V\|_2^2 \leq tr(S^2) - 2tr(S) + tr(I) = tr((S - I)^2) = \|S - I\|_2^2.$$

If A is invertible, then S is also invertible and the equality $\|S - V\|_2 = \|S - I\|_2$ implies we have the equality $|tr(SV)| = trS$, which we know is only possible when $V = e^{i\varphi}I$.

On the other hand, this equality also implies

$$2tr(S) = tr(SV^* + VS) \implies 2tr(S) = e^{-i\varphi}tr(S) + e^{i\varphi}tr(S)$$

Since S is invertible, we have that $tr(S) > 0$. This way, $\cos\varphi = 1 \implies \varphi = 2k\pi$ and then $V = I$, which finally implies $W = U$.

Problem 16. *Prove that if A is a normal operator (that is, $AA^* = A^*A$) and $A = SU$ is its polar decomposition then $SU = US$.*

Solution: $SU(SU)^* = AA^* = A^*A = (SU)^*SU$ Thus, $SUU^*S^* = U^*S^*SU$ Thus, $S^2 = U^*S^2U$. Since $S \geq 0$, we have that $S = \sqrt{S^2}$, thus taking the matrix square root on both sides of this last equation leads the result.

2.1 Singular value decomposition

The matrix S in the polar decomposition, being itself a symmetric matrix, can be further decomposed as $S = PDP^*$ using the spectral theorem. Since $S \geq 0$, it follows that the eigenvalues in the diagonal matrix D are non-negative. This way, there exists two orthogonal (unitary) matrices P and Q such that A can be "almost diagonalized" as $A = PDQ$. The elements in the diagonal D are called the *singular values* of A, and this is called the *singular value decomposition.*

Problem 17. *If A is an $n \times n$ normal matrix, show that the singular values $\sigma_1(A), \ldots, \sigma_n(A)$ of A are the absolute values of its eigenvalues: $|\lambda_1(A)|, \ldots, |\lambda_n(A)|$.*

Problem 18. *Prove the Courant-Fischer min-max formula for singular values:*

$$\sigma_i(A) = \inf_{dim(S)=n-i+1} \left(\sup_{x \in S} \frac{|Av|}{|v|} \right)$$

for all $1 \leq i \leq p$, where the supremum ranges over all subspaces of $\mathbb{C}^n$ of dimension i.