

Lecture 4

Paulo Sampaio

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1 Ordinary generating functions

Problem 1. If a_n is a sequence such that $a_{n+1} = 2a_n + 1$ and $a_0 = 0$, find an expression for a_n . Do the same for $a_{n+1} = 2a_n + n$ with $a_0 = 1$.

Problem 2. Prove the general term for the Fibonacci sequence using generating functions and show that $\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2$.

2 Cauchy product

Problem 3. If $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$, prove that the coefficient of x_n in the power series expansion of $A(x)B(x)$ is

$$c_k = \sum_{k=0}^n a_k b_{n-k}.$$

An interpretation of this theorem: If a_k counts all sets of size k of type S , and b_k counts all sets of size k of type T , then c_k counts all pairs of sets (S, T) where the total number of elements in both sets is k .

Problem 4. Prove that

$$\binom{m+n}{k} = \sum_{j=0}^k \binom{m}{j} \binom{n}{k-j}.$$

Problem 5 (Catalan numbers). Prove that the number of ways one can insert parentheses into a product of $n+1$ factors is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. Hint: use that $\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!} x^n$, for $|x| < 1$.

Problem 6 (Stirling numbers of the second kind). Determine the number of partitions of $[n] = \{1, 2, \dots, n\}$ into k classes.

3 Exponential generating functions

Problem 7 (Bell numbers). Determine the number of partitions of a set of n elements.

Problem 8. Determine a closed form expression for $B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$, the exponential generating functions for the Bell numbers b_n . Determine also the ordinary generating function for these numbers $B_o(x) = \sum_{n \geq 0} b_n x^n$.

Problem 9. Prove that the Bell numbers b_n in the last problem satisfy the recurrence

$$\begin{cases} b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} b_k & n \geq 1 \\ b_0 = 1 \end{cases}$$

Problem 10. A derangement of n letters is a permutation that has no fixed points. Find the number of derangements of n (distinct) letters.

4 Generating functions at the IMC

Problem 11 (IMC 1999). We throw a dice (which selects one of the numbers $1, 2, \dots, 6$ with equal probability) n times. What is the probability that the sum of values is divisible by 5?

Problem 12 (IMC 2003). Let $(a_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$a_{n+1} = \frac{1}{n+1} \sum_{k=0}^n \frac{a_k}{n-k+2}, \quad a_0 = 1.$$

Find the limit

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{a_k}{2^k}.$$

Problem 13 (IMC 2012). For every positive integer n , let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Also define $p(0) = 1$.

Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.