Lecture 5

Paulo Sampaio (paulo-elpidio.alves-sampaio@polytechnique.edu)

March 28, 2024

1 Basic asymptotics of integrals

Problem 1. *Find the asymptotic behavior as $A \to \infty$ of*

$$\int_A^{2A} (\ln x)^{-1}\, dx$$

Problem 2. *Find the asymptotic behavior as $A \to \infty$ of*

$$\int_1^{+\infty} e^{-x^p/A} x^{-1}\, dx \ (p > 0)$$

Problem 3. *Prove that, as $\varepsilon \to +0$,*

$$\textit{(a)} \int_0^1 x^{\varepsilon x}\, dx = 1 - \frac{\varepsilon}{4} + \frac{\varepsilon^2}{27} + O(\varepsilon^3)$$

$$\textit{(b)} \int_0^1 e^{-\varepsilon/\sqrt{x}}\, dx = 1 - 2\varepsilon - \varepsilon^2 \ln \varepsilon + O(\varepsilon^2)$$

2 Laplace method

Problem 4. *Find the asymptotics as $A \to +\infty$ of the following integrals:*

$$\textit{(a)} \int_0^1 (1 - x^p)^A\, dx \ (p > 0)$$

$$\textit{(b)} \int_0^{+\infty} (1 + x^p)^{-A}\, dx \ (p > 0)$$

$$\textit{(c)} \int_0^{\pi/2} x^p \cos^A x\, dx \ (p > -1).$$

Problem 5. *Suppose that $-\infty < a < b \le +\infty$, the function φ is positive and decreasing on $[a, b)$, $\int_a^b \varphi(x)\, dx < +\infty$, and $\varphi(a) - \varphi(x) \sim_{x \to a+0} C \cdot (x - a)^{1/p}$, where $C, p > 0$. Prove that*

$$\Phi(A) = \int_a^b \varphi^A(x)\, dx \sim_{A \to +\infty} \left(\frac{\varphi(a)}{AC}\right)^p \Gamma(1 + p)\varphi^A(a).$$

In particular, $\Phi(A) \sim 1/(AC)$ for $p = 1$, and $\Phi(A) \sim \frac{1}{2}\sqrt{\pi/(AC)}$ for $p = 1/2$.

Problem 6. *Suppose that $-\infty < a < b \le +\infty$, the function φ is positive and decreasing on $[a, b)$, $\int_a^b \varphi(x)\, dx < +\infty$, and $\lim_{x \to a+0}(1 - \varphi(x))/(x - a)^{1/p} = +\infty$ for some $p > 0$. Prove that $\int_a^b \varphi^A(x)\, dx = o(A^{-p})$ as $A \to +\infty$.*

3 Asymptotics of unknown quantities

Problem 7. *Let x_n be the root of the equation $x = \tan x$ lying in the interval $(\pi n, \pi(n + 1)), n \in \mathbb{N}$. Prove that*

$$x_n = \pi n + \frac{\pi}{2} - \frac{1}{\pi n} + O\left(\frac{1}{n^2}\right).$$

4 Asymptotic of integrals of high dimension

Problem 8. *Let P_n be the normalized Lebesgue measure on the sphere $S^{n-1}(\sqrt{n})$. Prove that the coordinates on this sphere "are asymptotically distributed according to a normal law", that is, for any $a, b \in \mathbb{R}$, with $a < b$*

$$\lim_{n\to\infty} P_n\{(x_1, \dots, x_n) \in S^{n-1}(\sqrt{n}) | a < x_n < b\} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2}\, dt.$$

Problem 9. *Prove that*

$$\frac{1}{\sqrt{n}} \int_{[0,1]^n} \|x\|\, dx \to_{n\to\infty} \frac{1}{\sqrt{3}}$$

Problem 10. *Prove that, for any $f \in C([0,1])$,*

$$\int_{[0,1]^n} f(\sqrt[n]{x_1 \dots x_n})\, dx \to f(e^{-1}).$$

Problem 11. *Prove that, for any $f \in C(\mathbb{R}^n)$ with $\sup |f| < \infty$,*

$$\int_{\mathbb{R}^n} f\left(\frac{|x_1| + \cdots + |x_n|}{n}\right)\, d\gamma_n(x) \to f\left(\sqrt{\frac{2}{\pi}}\right),$$

where γ_n is the standard Gaussian measure over $\mathbb{R}^n$, that is, the measure with density $(2\pi)^{-n/2}e^{-\|x\|^2/2}$.