Lecture 6

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1 The Binet-Cauchy Formula

Problem 1. *Let A and B be matrices of size $n \times m$ and $m \times n$, respectively, and $n \leq m$. Then*

$$\det(AB) = \sum_{1 \leq k_1 < k_2 < \cdots < k_n \leq m} \det(A_{k_1 \ldots k_n}) \det(B^{k_1 \ldots k_n}),$$

where $A_{k_1 \ldots k_n}$ is the sub-matrix obtained from the columns of A whose numbers are $k_1, \ldots, k_n$ and $B^{k_1 \cdots k_n}$ is the sub-matrix obtained from the rows of B whose numbers are $k_1, \ldots, k_n$.

2 Applications

2.1 Spanning trees enumeration

Problem 2 (Kirchhoff's matrix tree theorem)**.** *For a given connected graph G with n labeled vertices, let L be the Laplacian matrix of G and denote $L[i]$ the matrix L with the ith row and column removed. Prove that the number of spanning trees of G is $\det L[i]$, for any admissible i.*

Problem 3 (Cayley's formula)**.** *Prove that the number of trees on n labeled vertices is n^{n-2}.*

2.2 Multiplicativity properties of compound matrix

For a matrix A of size $m \times n$ we can also consider the matrix whose elements are the r-th order minors, arranged in lexicographic order. For example, if A is a 3×3 matrix, then

$$C_2(A) = \begin{pmatrix} \det(A^{1,2}_{1,2}) & \det(A^{1,2}_{1,3}) & \det(A^{1,2}_{2,3}) \\ \det(A^{1,3}_{1,2}) & \det(A^{1,3}_{1,3}) & \det(A^{1,3}_{2,3}) \\ \det(A^{2,3}_{1,2}) & \det(A^{2,3}_{1,3}) & \det(A^{2,3}_{2,3}) \end{pmatrix}.$$

This $C_r(A)$ is called the r-th compound matrix of A.

Problem 4. *Prove that $C_r(AB) = C_r(A)C_r(B)$.*

Problem 5 (Jacobi's theorem)**.** *Let $A = (a_{ij})_{ij}$, $(\operatorname{adj} A)^T = (A_{ij})_{ij}$, $1 \leq p \leq n$, $\sigma = \begin{pmatrix} i_1 & \ldots & i_n \\ j_1 & \ldots & j_n \end{pmatrix}$ an arbitrary permutation. Prove that*

$$\begin{pmatrix} A_{i_1 j_1} & \ldots & A_{i_1 j_p} \\ \vdots & \ldots & \vdots \\ A_{i_p j_1} & \ldots & A_{i_p j_p} \end{pmatrix} = (-1)^\sigma \begin{vmatrix} a_{i_{p+1}, j_{p+1}} & \ldots & a_{i_{p+1}, j_n} \\ \vdots & \ldots & \vdots \\ a_{i_n, j_{p+1}} & \ldots & a_{i_n, j_n} \end{vmatrix} \cdot |A|^{p-1}.$$

2.3 Smith normal form

Let A be a matrix whose elements are integers or polynomials and let $f_k(A)$ be the greatest common divisor of minors of order k of A. The formula for determinant expansion with respect to a row indicates that f_k is divisible by f_{k-1}.

Problem 6. *If $A' = BAC$, where B and C are unity matrices, prove that $f_k(A') = f_k(A)$ for all admissible k.*

Problem 7 (Smith normal form)**.** *Prove that, for any matrix A of size $m \times n$ there exist unity matrices B and C such that $BAC = diag(g_1, g_2, \ldots, g_p, 0, \ldots, 0)$, where g_{i+1} is divisible by g_i.*

The matrix $diag(g_1, g_2, \ldots, g_p, 0, \ldots, 0)$ is called the Smith normal form of A.