

Lecture 7 - Spectral Graph Theory

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1 Spectrum of a graph

For a graph G we have defined in last class the *Laplacian* matrix:

$$L(u, v) = \begin{cases} d_v & \text{if } u = v, \\ -1 & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Today, we will be interested in a closely related matrix, called the *normalized Laplacian*:

$$\mathcal{L}(u, v) = \begin{cases} 1 & \text{if } u = v, \\ -\frac{1}{\sqrt{d_u d_v}} & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

For us, the spectrum of a graph will be the spectrum of $\mathcal{L}$, noted $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$. We usually note $\lambda_G = \lambda_1$.

Exercise 1. Prove that for a graph G on n vertices we have $\sum_i \lambda_i \leq n$, with equality if and only if G has no isolated vertices.

Exercise 2. Prove that if G is not a complete graph, then $\lambda_1 \leq 1$.

Exercise 3. Prove that the spectrum of a graph is the union of the spectra of its connected components. Prove that if G is connected, then $\lambda_1 > 0$. Deduce that if $\lambda_i = 0$ and $\lambda_{i+1} \neq 0$, then G has exactly $i+1$ connected components.

2 Bipartite graphs

A graph (V, E) is called bipartite if one can divide the set of vertices into two disjoint sets V_1 and V_2 such that each edge in E connects a vertex in V_1 to a vertex in V_2 .

Exercise 4. Prove that for all $i \leq n-1$ we have $\lambda_i \leq 2$, with $\lambda_{n-1} = 2$ if and only if a connected component of G is bipartite and non-trivial.

Exercise 5. Prove that the following statement are equivalent:

- (i) G is bipartite
- (ii) G has $i+1$ connected components and $\lambda_{n-j} = 2$ for $1 \leq j \leq i+1$.
- (iii) For each λ_i , the value of $2 - \lambda_i$ is also an eigenvalue of G .

3 Weighted graphs

A weighted undirected graph $G = (V, E)$ is a graph with an associated weight function $w : V \times V \rightarrow \mathbb{R}$ such that:

1. $w(u, v) \geq 0, \forall v \in V$
2. $w(u, v) = w(v, u), \forall u, v \in V$.

We define the similar matrices L and $\mathcal{L}$ for weighted graphs:

$$L(u, v) = \begin{cases} d_v - w(v, v) & \text{if } u = v, \\ -w(u, v) & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases} \quad \mathcal{L}(u, v) = \begin{cases} 1 - \frac{w(v, v)}{d_v} & \text{if } u = v, \\ -\frac{w(u, v)}{\sqrt{d_u d_v}} & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

A contraction of a graph G is formed by identifying two distinct vertices, say u and v , into a single vertex v^* . The new weight function is defined as

$$\begin{cases} w'(x, v^*) = w(x, u) + w(x, v), & \text{if } x \sim u \text{ or } x \sim v \\ w'(v^*, v^*) = w(u, u) + w(v, v) + 2w(u, v) \\ w'(x, y) = w(x, y), & \text{otherwise} \end{cases}$$

Exercise 6. Prove that if H is formed by contractions of a graph G , then $\lambda_G \leq \lambda_H$.

4 The Cheeger constant

For a subset of vertices S , define the volume of S as the sum of the degrees of the vertices in S : $\text{Vol } S = \sum_{x \in S} d_x$.

The edge boundary of S is the set of all edges with just one endpoint in S :

$$\partial S = \{\{u, v\} \in E(G) : u \in S \text{ and } v \notin S\}.$$

The Cheeger constant of a graph is then defined as

$$h_G = \min_S \frac{|\partial S|}{\min(\text{Vol } S, \text{Vol } \bar{S})},$$

where $\bar{S}$ is the complement of S .

Exercise 7. Prove that $2h_G \geq \lambda_1$.

Remark. Exercise 7 is the first part of the so called Cheeger inequality. In fact with more work we can show that if the graph is connected,

$$\frac{h_G^2}{2} < \lambda_1 \leq 2h_G.$$

or even that

$$1 - \sqrt{1 - h_G^2} < \lambda_1 \leq 2h_G.$$

This way we see that the λ_1 eigenvalue measures the connectivity of the graph.